MAT 1330, Fall 2012 Assignment 1

Due Friday September 28, 3:00pm at the beginning of class.

Late assignments will not be accepted; nor will unstapled assignments.

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DGD (circle one): 1   2   3   4

Student Name ______________________ Student Number ______________________

By signing below, you declare that this work was your own and that you have not copied from any other individual or other source.

Signature ______________________

QUESTION 1. Consider the following data for monthly temperature of Montréal in 2010.

Month	Avg. Temp. (C°)
Jan	-6.4
Feb	-4.7
Mar	3.3
Apr	9.5
May	15.7
Jun	18.4
Jul	23
Aug	20.9
Sep	16.3
Oct	8.3
Nov	2.4
Dec	-5.8

We will model the temperature at time t during the year 2010 in Montréal using the general formula for oscillations:

$$\text{Temp}(t) = A + B\cos\left(\frac{2\pi}{T}(t-\phi)\right).$$

(Here the time unit is months)

a) Find the simple average by using only the minimum and maximum of the data set.
Answer:
Let A denote the simple average. $A = \dfrac{-6.4+23}{2} = 8.3$

b) Find the amplitude of the data with respect to the simple average.

Answer:

Let B denote the amplitude. $B = \dfrac{23+6.4}{2} = 14.7$

c) What is the period and phase shift?

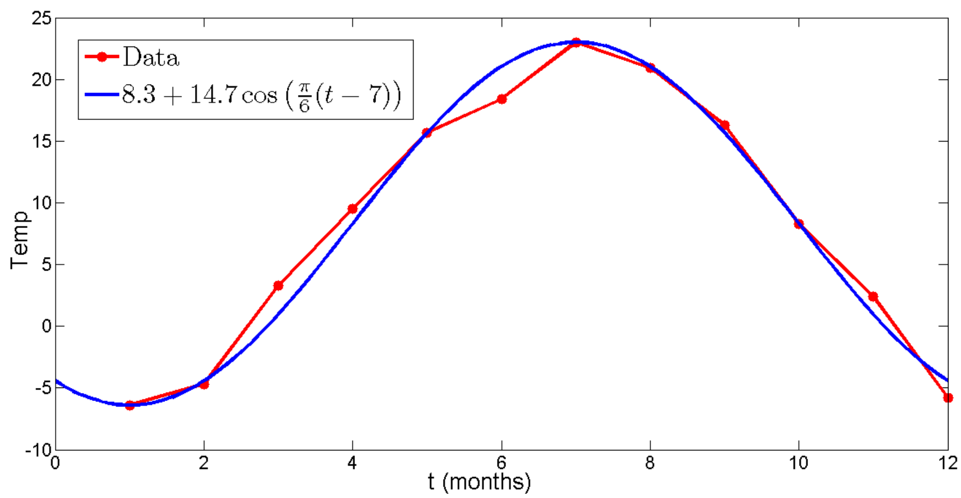
Answer:
Period: $T = 12$. If we take the January average temperature to occur at $t = 1$, then the phase shift is $\phi = 7$. If we take January average temperature to occur at $t = 0$, then the phase shift is $\phi = 6$

d) Write down the cosine function that models the temperature data. (Use parts a-c)

Answer:
If $\phi = 7$, $\text{Temp}(t) = 8.3 + 14.7\cos\left(\frac{\pi}{6}(t-7)\right)$. If $\phi = 6$, $\text{Temp}(t) = 8.3 + 14.7\cos\left(\frac{\pi}{6}(t-6)\right)$.

e) Graph the cosine functions obtained in part d with the data.

Answer:



If $\phi = 6$, the graph and the data would be shifted to the left 1 unit.

QUESTION 2. A new rule has been approved at the University of Ottawa. The rule stipulates that precisely 1/4 of all university students must leave at the end of each summer (and the remaining students must stay) and exactly 10,000 new students will be added at the same time.

a) What is the discrete-time dynamical system that gives the number of students at UO in a given year?

Answer:
The discrete-time dynamical system that gives the number of UO students is:

$$S_{t+1} = \frac{3}{4}S_t + 10,000$$

b) The number of students at UO is now $33,917$. How many students will there be two years later (round your answer)?
Answer:

$$
\begin{aligned}
S_1 &= \frac{3}{4}(33,917) + 10,000 \\
S_1 &= 35,437.75 \\
S_2 &= \frac{3}{4}(35,437.75) + 10,000 \\
S_2 &= 36,578
\end{aligned}
$$

c) Write down the updating function of the dynamical system.

Answer:
$f(S_t) = \frac{3}{4}S_t + 10,000.$
d) Find all equilibrium points of the dynamical system.

Answer:

$$
\begin{aligned}
S &= \frac{3}{4}S + 10,000 \\
\frac{1}{4}S &= 10,000 \\
S &= 40,000
\end{aligned}
$$

e) Find a formula for the solution to the dynamical system for initial condition $S_0 = 60,000$.
Answer:
Method 1: Consider $S_t = a \cdot (0.75)^t + b$.

$$
\begin{aligned}
S_0 &= a\left(\frac{3}{4}\right)^0 + b \\
\Rightarrow 60,000 &= a + b
\end{aligned}
$$

$$
\begin{aligned}
S_1 &= a\,(0.75)^1 + b \\
\text{Also, } S_1 &= 0.75 \cdot 60,000 + 10,000 \\
S_1 &= 55,000 \\
\Rightarrow 55,000 &= 0.75a + b \\
\Rightarrow 55,000 &= 0.75a + 60,000 - a \\
\Rightarrow 0.25a &= 5,000 \\
\Rightarrow a &= 20,000 \\
\Rightarrow b &= 40,000
\end{aligned}
$$

$$
\Rightarrow S_t = 20,000 \cdot (0.75)^t + 40,000
$$

Check solution:

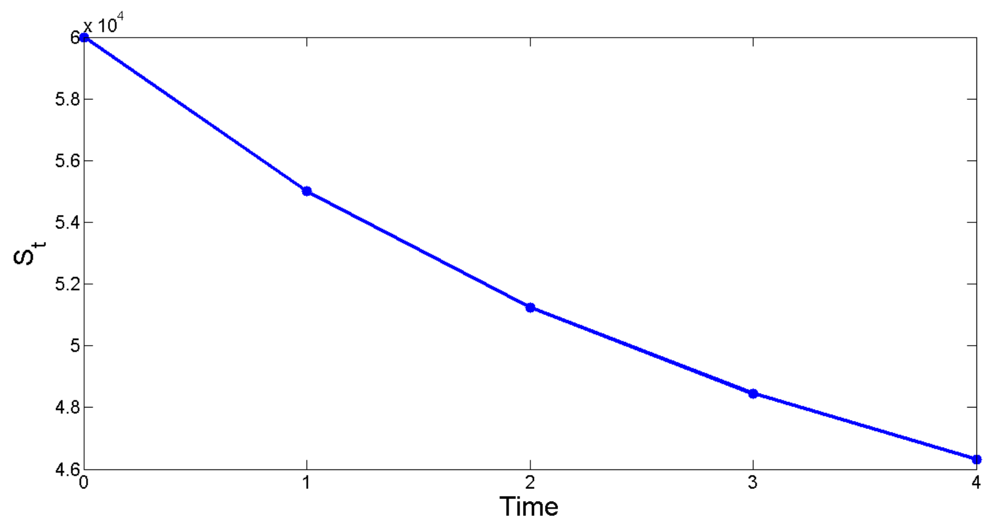
$$S_{t+1} = 0.75 \cdot (20,000 \cdot (0.75)^t + 40,000) + 10,000$$
$$S_{t+1} = 20,000 \cdot (0.75)^{t+1} + 0.75 \cdot 40,000 + 10,000$$
$$S_{t+1} = 20,000 \cdot (0.75)^{t+1} + 40,000$$

Method 2:

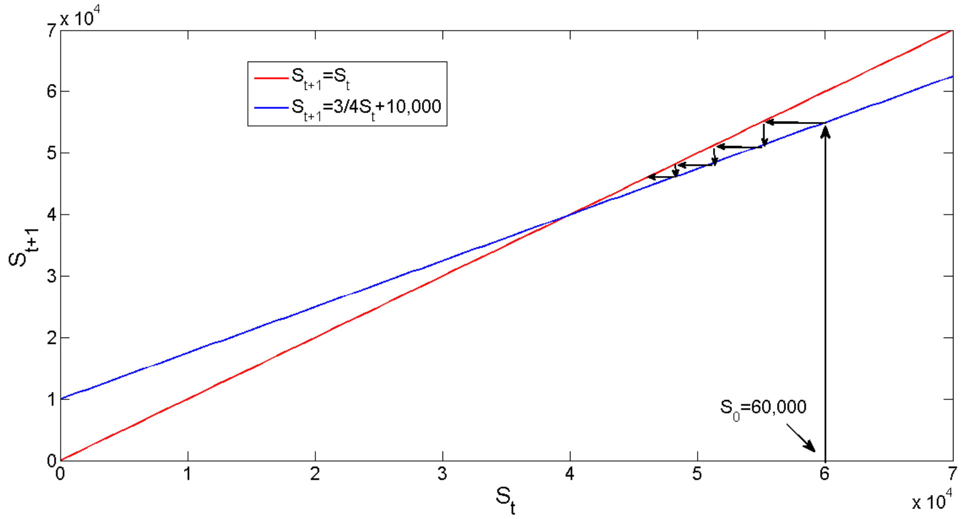
$$\begin{aligned}
S_1 - 40,000 &= 0.75S_0 - 30,000 = 0.75 \cdot 60,000 - 30,000 \\
&= 0.75 \cdot 60,000 - 0.75 \cdot 40,000 \\
&= 0.75(60,000 - 40,000) \\
&= 0.75(20,000) \\
S_2 - 40,000 &= 0.75S_1 - 30,000 = 0.75(0.75(20,000) + 40,000) - 30,000 \\
&= (0.75)^2 \cdot 20,000 + 0.75 \cdot 40,000 - 30,000 \\
&= (0.75)^2 \cdot 20,000 \\
&\vdots \\
S_t - 40,000 &= (0.75)^t \cdot 20,000 \\
\Rightarrow S_t &= 20,000 \cdot (0.75)^t + 40,000
\end{aligned}$$

f) Draw the solution of the dynamical system with $S_0 = 60,000$. (Four points are enough.)

Answer:

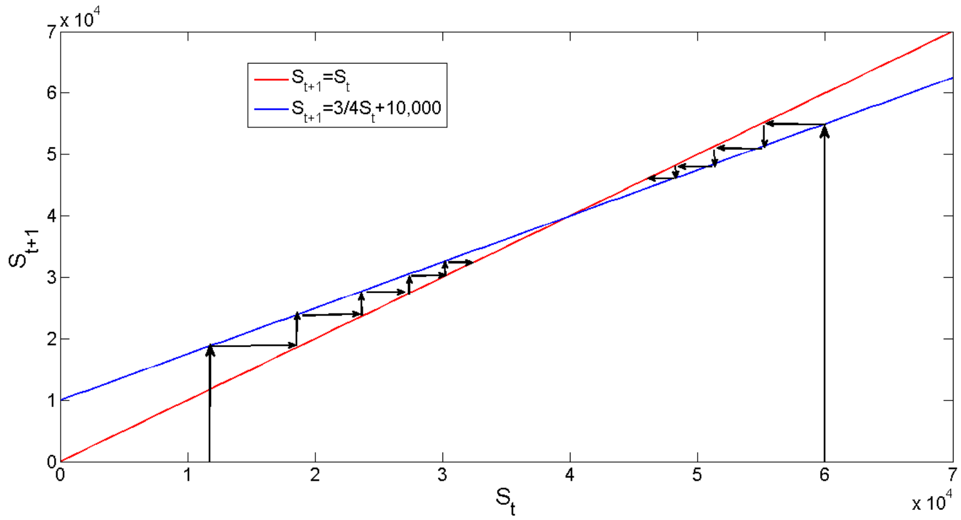


g) Draw the cobweb diagram of the dynamical system with $S_0 = 60,000$. (Four iterations are enough.)
Answer:



h) Determine the stability of the equilibrium point using the cobweb diagram.

Answer:



Question 3. Consider the following discrete-time model for the population of a species:

$$N_{t+1} = \frac{rN_t}{1 + N_t}$$

where r is a constant. For what values of r does the model have a positive equilibrium?
Answer:

Find equilibria:

$$\begin{aligned}
N &= \frac{rN}{1+N} \\
N(1+N) - rN &= 0 \\
N(N-(r-1)) &= 0 \\
&\Rightarrow N = 0, N = r-1
\end{aligned}$$

$r-1$ is positive when $r > 1$. Therefore $r > 1$ is the answer.

QUESTION 4. A population of rabbits is growing at a rate of 10% per year. Write down the discrete dynamical system which describes the evolution of the rabbit population. If the population is initially 1000, how many years will it take for the population to exceed 100000?
Answer:
$R_{t+1} = 1.1R_t$ and $R_0 = 1000$.

$$\begin{aligned}
100,000 &= (1.1)^t(1000) \\
100 &= (1.1)^t \\
\ln 100 &= t \ln 1.1 \\
\frac{\ln 100}{\ln 1.1} &= t
\end{aligned}$$

$$t = \frac{\ln 100}{\ln 1.1} \text{ years}$$

It will take $\frac{\ln 100}{\ln 1.1}$ years.

Other answers:
It will take 48.3177 years
It will take 49 years.