Solutions

MATH 1013 Applied Calculus Section A Fall 2012

Test 1/ October 3, 2012

Student Name: **ID-No.:**

You have 50 minutes to solve the following problems: Show your complete work.

Permitted aids: 1) calculator and 2) help sheet of standard letter size with notes and formulas.

Problem 1) **(16 points)**

Find $f \circ g$ and $g \circ f$ together with their domains:

a) $f(x) = x^3$, $x > 0$; $g(x) = 1/(x+2)$, $x \neq -2$;

b) $f(x) = \ln(x)$, $x>0$; $g(x) = \sin(x)$, $-\pi/2 < x \leq \pi/2$.

a) $f \circ g(x) = f(g(x)) = f\left(\frac{1}{x+2}\right) = \frac{1}{(x+2)^3}$

$D(f \circ g) = \{x \in \mathbb{R} \mid x > -2\}$

$g \circ f(x) = g(f(x)) = g(x^3) = \frac{1}{x^3+2}$

$D(g \circ f) = \{x \in \mathbb{R} \mid x > 0\}$

b) $f \circ g(x) = \ln(\sin(x))$; $D(f \circ g) = \{x \mid 0 < x \leq \frac{\pi}{2}\}$

$g \circ f(x) = \sin(\ln(x))$; $D(g \circ f) = \{x \in \mathbb{R} \mid e^{-\pi/2} < x \leq e^{\pi/2}\}$

$= [0.208, 4.81]$

Problem 2) **(16 points)**

a) Suppose $A(t)=2^{kt}$. Find a value of k so that $A(5)=17$.

b) Solve the equation $\log_3(x^2 + 1) = 2$.

c) Use the algebraic rules of the logarithm to

i) expand $\log_4\left(\frac{x^6 y^5}{(x+y)^8}\right)$ into several, simpler logarithms.

ii) compress $3\log(x) - \log(y) - 7\log(z)$ into a single logarithm.

d) Carbon-14 is used for radiometric dating of organic matter (such as archaeological traces) If N_0 is the number of carbon-C atoms in the tissue (at the time of death) the number of carbon-14 atoms after t years, due to radioactive decay, is given by

$$N(t)= N_0(1/2)^{t/5730} .$$

How old is the object under investigation if the concentration of carbon-14 has decreased to 5% of its original value?

a) $2^{5k} = 17 \Rightarrow \ln(2^{5k}) = 5k \cdot \ln(2) = \ln(17)$

$\Rightarrow k = \dfrac{\ln(17)}{5 \cdot \ln(2)} = 0.817$

b) $\log_3(x^2+1) = 2 \Rightarrow 3^2 = x^2+1 \Rightarrow x^2 = 8 \Rightarrow x = \pm\sqrt{8}$

c) i) $6\log_4(x) + 5\cdot\log_4(y) - 8\cdot\log_4(x+y)$

ii) $\log\left(\dfrac{x^3}{y \cdot z^7}\right)$

d) $N(t) = 0.05 \cdot N_0 = N_0 \cdot \left(\frac{1}{2}\right)^{t/5730}$

simplify and logarithmize: $\ln(0.05) = \ln\left(\left(\frac{1}{2}\right)^{t/5730}\right)$

$\Rightarrow t = 5730 \cdot \dfrac{\ln(0.05)}{\ln(1/2)} = 24{,}764.6$ years

Problem 3: **(24 points)**

State whether false or true:

i) The cosine is an even function. TRUE

ii) All function are either even or odd. FALSE

iii) For any function f(x), 3f(x-2)+2 shifts the graph horizontally to the right, compresses the graph vertically by the factor 3 and shifts the graph upwards by two units.

FALSE

iv) Suppose f(t)= 8^t. Then f(t+h) = f(t)f(h). TRUE

v) The slope of the tangent to the graph of a function in a particular point (x,f(x)) is given by the difference quotient $\Delta y/\Delta x$, for any increment Δx.

FALSE

vi) The derivative of a function in a point (x,f(x)) is given, geometrically, as the slope of the tangent to the graph in that point.

TRUE

vii) The tangens function is an even function and defined everywhere.

FALSE

viii) For an angle θ, the $\sin(\theta)$ function gives the ratio of the opposite side and the hypotenuse: $\sin(\theta)$ is an odd function and is defined for all θ.

TRUE

Problem 4) **(24 points)**

Calculate the limit if it exists, or state DNE (does not exist):

a) $\lim_{t \to 2}(5t^2 - 3t + 1) = \lim_{t \to 2}(5t^2) - 3\lim_{t \to 2}(t) + 1$
$= 20 - 6 + 1 = 15$

b) $\lim_{t \to 0}(3t + 1)\cos(t)$
$= \lim_{t \to 0}(3t+1) \cdot \lim_{t \to 0}\cos(t)$
$= 1 \cdot 1 = 1$

c) $\lim_{t \to 3} \frac{\sqrt{19 - t} - 4}{t - 3}$

d) $\lim_{\phi \to 0} \frac{\sin(8\phi)}{\cos(7\phi)} = \frac{\lim_{\phi \to 0} \sin(8\phi)}{\lim_{\phi \to 0} \cos(7\phi)} = \frac{0}{1} = 0$

e) The ozone depletion potential (ODP) of a chemical is sometimes expressed as the ratio of its integrated chlorine loading to that of chlorofluorocarbon CFC-11. The ODP of NH_4ClO_4 (a chemical used in rocket fuel) is

$$\lim_{t \to \infty} \frac{0.318 e^{-t/3}}{2.33(1 + 0.16e^{-t/3} - 1.11e^{-t/46.5} - 0.05e^{-t/1.75})}.$$

Calculate the ODP of NH_4ClO_4 and determine whether it's better or worse for the ozone layer than CFC-11.

with the corrected numerator:

$$\lim_{t \to \infty} \left(\frac{0.318 \cdot (1 - e^{-t/3})}{2.33 \cdot (1 + 0.16 \cdot e^{-t/3}) - 1.11e^{-t/46.5} - 0.05 e^{-t/1.75})} \right)$$

$$= \frac{0.318 \cdot (1 - 0)}{2.33 \cdot (1 + 0 - 0 - 0)} = 0.1365:$$ NH_4ClO_4 is better by a factor of approx. 7.

4)c) $$\lim_{t \to 3} \frac{\sqrt{19-t} - 4}{t - 3} = \lim_{t \to 3} \frac{(\sqrt{19-t} - 4) \cdot (\sqrt{19-t} + 4)}{(t-3) \cdot (\sqrt{19-t} + 4)} =$$

$$= \lim_{t \to 3} \frac{19 - t - 16}{(t-3) \cdot (\sqrt{19-t} + 4)} = \lim_{t \to 3} \frac{\cancel{3-t}}{\cancel{(t-3)} \cdot (\sqrt{19-t}+4)} = \frac{-1}{4+4}$$
$$= -0.125.$$

Problem 5: **(20 points)**

The points (4,2) and (4+h, $\sqrt{4+h}$) are on the graph of f when f(t)=$\sqrt{t}$. denote the slope of the secant through these two points by m(h).

Determine a formula for m(h) and calculate its limit for h→0. What does this limit represent for the graph of f?

$$m(h) = \frac{f(t+h) - f(t)}{h}$$

$$= \frac{\sqrt{4+h} - 2}{h}$$

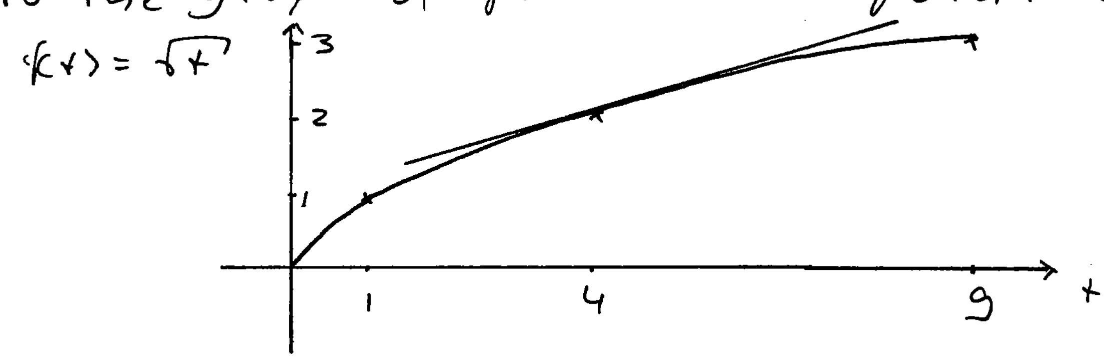
$$\lim_{h \to 0} m(h) = \lim_{h \to 0} \frac{\sqrt{4+h} - 2}{h}$$

$$= \lim_{h \to 0} \frac{(\sqrt{4+h} - 2)(\sqrt{4+h} + 2)}{h \cdot (\sqrt{4+h} + 2)}$$

$$= \lim_{h \to 0} \frac{4+h-4}{h \cdot (\sqrt{4+h} + 2)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{4+h} + 2} = \frac{1}{4}.$$

This limit represents the slope of the tangent to the graph of f(t) in the point (4, 2).



Bonus question (10 points)

For the function shown below, $\lim_{t \to 10.5^-} f(t) = 3.$

$$f(t) = \begin{cases} \sqrt{t-1.5} & \text{when } 1.5 < t < 10.5 \\ -2t+4 & \text{when } t > 10.5. \end{cases}$$

Use the ε-δ definition for the (one-sided) limit and find an appropriate δ when ε = 0.25.

$\lim_{t \to 10.5^-} f(t) = 3$:

For every $\varepsilon > 0$ there exists a $\delta > 0$ such that

for all t with $10.5 - \delta < t < 10.5$

(**) $|f(t) - 3| < \varepsilon$: $|\sqrt{t-1.5} - 3| < \varepsilon$.

Find δ for $\varepsilon = 0.25$:

$|\sqrt{t-1.5} - 3| < 0.25$: expand into two-sided inequality :

$\Leftrightarrow \quad 3 - 0.25 < \sqrt{t-1.5} < 3 + 0.25$

$\Leftrightarrow \quad (3-0.25)^2 < t - 1.5 < (3+0.25)^2$

$\Leftrightarrow \quad 9.0625 < t < 12.0625$ (*)

Choose $\delta = 1$: Then for all t with

$9.5 < t < 10.5$

inequality (*) and therefore (**) is fulfilled.