University of Toronto Scarborough
Department of Computer & Mathematical Sciences

MATA30: Calculus I - Midterm Test

Examiners: Sophie Chrysostomou

Date: Wednesday, October 31, 2012
Duration: 110 minutes

DO NOT OPEN THIS BOOKLET UNTIL INSTRUCTED TO DO SO.

FAMILY NAME: ______________________________

GIVEN NAMES: ______________________________

STUDENT NUMBER: ______________________________

DAY AND TIME OF YOUR TUTORIAL: ______________________________

SIGNATURE:______________________________

CIRCLE THE NAME OF YOUR TEACHING ASSISTANT:

Fazle Chowdhury	Fangzhou Guo	Hnidec Samantha
Manaal Hussain	Mike Milijanovic	Ushya Shanmugarajah
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Junsheng Wu		

NOTES:

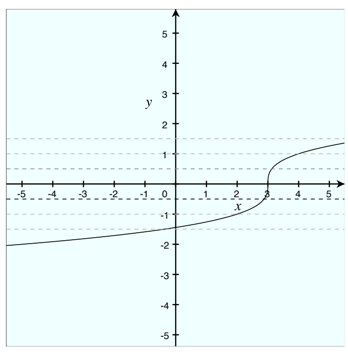
- No calculators, or any electronic aid is permitted at the test room.
- No cell phones are permitted.
- No books, notebooks or scrap paper are permitted near your examination table.
- There are 12 numbered pages in the test. It is your responsibility to ensure that, at the start of the test, this booklet has all its pages. The last two pages (11 and 12) are empty.
- Please leave all the pages of this booklet stapled. Do not remove any pages.
- Answer all questions in the space provided. Show your work and justify your answers for full credit.

FOR MARKERS ONLY	
1	/ 10
2	/10
3 a	/ 5
3 b	/ 5
3 c	/ 5
3 d	/ 5
3 e	/ 5
4	/ 5
5	/ 5
6	/ 10
7a	/ 5
7b	/ 5
8a	/ 5
8b	/ 5
8c	/ 5
8d	/ 5
8e	/ 5
TOTAL	**/100**

1. [**10 mark**] Let $f(x) = \sqrt[3]{x-3}$

   (a) [**2 marks**] Find the domain and range of f.
   Solution:
   The domain of f is $\mathbb{R}$.
   The range of f is $\mathbb{R}$.

   (b) [**1 mark**] Show that f is invertible.
   Solution:
   If $x_1 \neq x_2$, then $x_1 - 3 \neq x_2 - 3$ and therefore $\sqrt[3]{x_1 - 3} \neq \sqrt[3]{x_2 - 3}$. This shows that f is one to one and therefore invertible.
   Or: by using the horizontal line test on the graph of f below, we see that no horizontal line crosses the graph of f more than once, so f is one to one and therefore invertible.



   (c) [**3 marks**] Find f^{-1}, the inverse of f, and give its domain and range.
   Solution:
   Let $y = \sqrt[3]{x-3}$. Then:

$$\begin{aligned} y^3 &= x - 3 \\ y^3 + 3 &= x \end{aligned}$$

   Switching x and y we get that the inverse function $y = f^{-1}$ is given by $y = x^3 + 3$.

   The domain of f^{-1} is the range of f which is $\mathbb{R}$.

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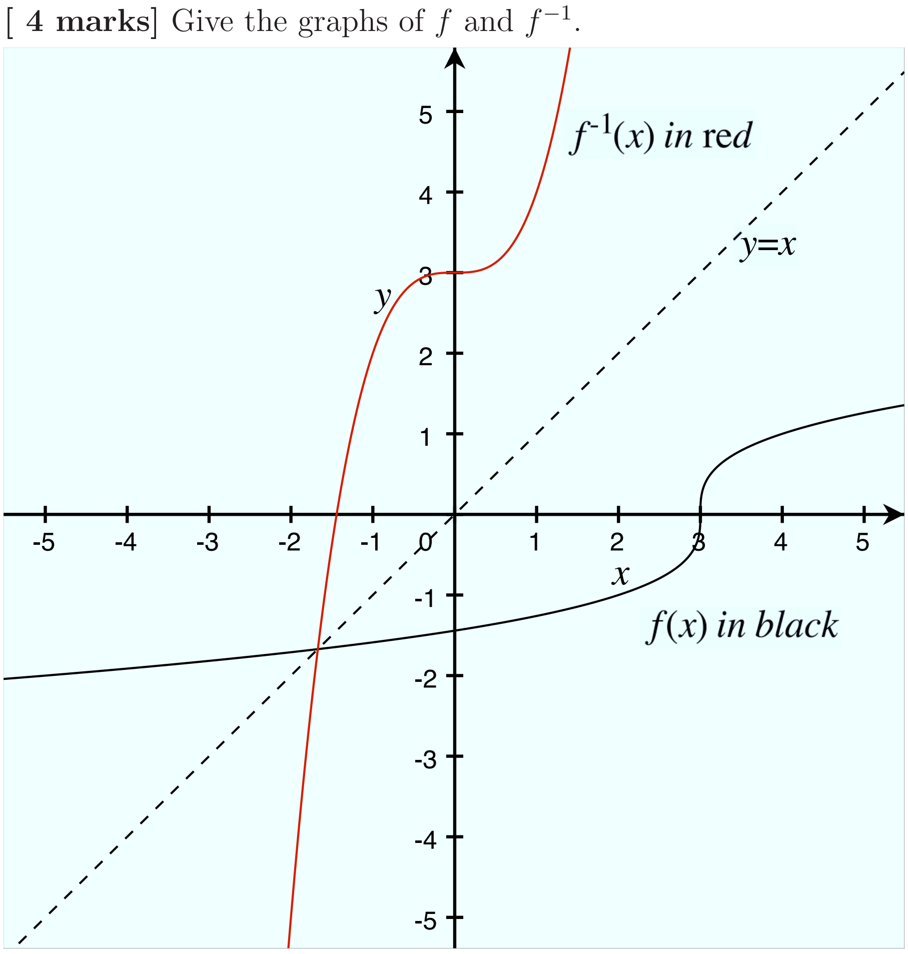
   (d) [**4 marks**] Give the graphs of f and f^{-1}.

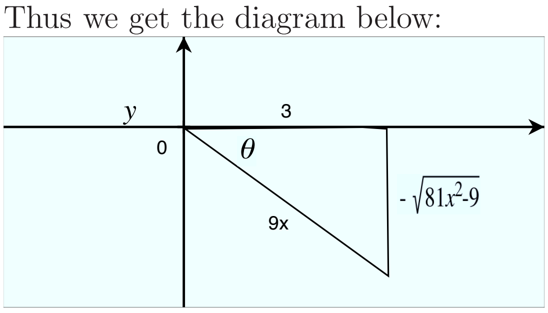


2. (a) [**4 marks**] If $\sec\theta = \dfrac{9x}{3}$ for $\theta \in \left(\dfrac{3\pi}{2}, 2\pi\right)$ and for some $x > 0$, find the following trigonometric ratios:

$\sin x = -\dfrac{\sqrt{81x^2-9}}{9x}$

$\cot x = -\dfrac{3}{\sqrt{81x^2-9}}$

Since $\theta \in \left(\dfrac{3\pi}{2}, 2\pi\right)$ the angle is in the fourth quadrant. Since $\sec\theta = \dfrac{9x}{3}$, then we could have the adjacent side have length 3 and the hypotenuse have length $9x$. Thus we get the diagram below:



(b) [**4 marks**] Prove the identity:

$$\frac{1}{1-\cos x} + \frac{1}{1+\sin x} = \csc^2 x \sec^2 x + \csc x \cot x - \sec x \tan x$$

Solution:

$$\begin{aligned}
RHS &= \frac{1}{1-\cos x} + \frac{1}{1+\sin x} = \frac{1+\cos x}{(1-\cos x)(1+\cos x} + \frac{1-\sin x}{(1+\sin)(1-\sin x)} \\
&= \frac{1+\cos x}{1-\cos^2 x} + \frac{1-\sin x}{1-\sin^2 x} = \frac{1+\cos x}{\sin^2 x} + \frac{1-\sin x}{\cos^2 x} \\
&= \frac{\cos^2 x + \cos^3 x + \sin^2 x - \sin^3 x}{\sin^2 x \cos^2 x} = \frac{1+\cos^3 x - \sin^3 x}{\sin^2 x \cos^2 x} \\
&= \frac{1}{\sin^2 x \cos^2 x} + \frac{\cos^3 x}{\sin^2 x \cos^2 x} - \frac{\sin^3 x}{\sin^2 x \cos^2 x} = \csc^2 x \sec^2 x + \frac{\cos x}{\sin^2 x} - \frac{\sin x}{\cos^2 x} \\
&= \csc^2 x \sec^2 x + \csc x \cot x - \sec x \tan x = LHS
\end{aligned}$$

Thus:

$$\frac{1}{1-\cos x} + \frac{1}{1+\sin x} = \csc^2 x \sec^2 x + \csc x \cot x - \sec x \tan x$$

(c) [**2 marks**] Find all $x \in \mathbb{R}$ satisfying: $\log_x(12 - 2x - x^2) = 2$.

Solution: Since x is a base of a logarithmic function, then $x > 0$ and by the definition of the logarithmic function:

$$\log_x(12 - 2x - x^2) = 2 \iff x^2 = 12 - 2x - x^2$$

We need to find the solution to $x^2 = 12 - 2x - x^2$ or $2x^2 + 2x - 12 = 0$

$$0 = 2x^2 + 2x - 12 = 2(x^2 + x - 6) = 2(x-2)(x+3)$$

The roots of the quadratic are $x = 2$ or $x = -3$ however, since x must be positive then the only solution is $x = 2$.

3. [**25 marks total**] Evaluate each of the following limits, else explain why the limit does not exist. Justify your answer. Do not use L'Hôspital's rule.

(a) $\lim_{x\to 5} \dfrac{x^2-3x-10}{x^2-25}$

Solution:

$$\begin{aligned}\lim_{x\to 5} \frac{x^2-3x-10}{x^2-25} &= \lim_{x\to 5} \frac{(x-5)(x+2)}{(x-5)(x+5)} \\ &= \lim_{x\to 5} \frac{(x+2)}{(x+5)} \\ &= \frac{7}{10}\end{aligned}$$

(b) $\lim_{x\to 0} \dfrac{x}{\sqrt{1+x}-\sqrt{1-x}}$

Solution:

$$\begin{aligned}\lim_{x\to 0} \frac{x}{\sqrt{1+x}-\sqrt{1-x}} &= \lim_{x\to 0} \frac{x}{(\sqrt{1+x}-\sqrt{1-x})} \cdot \frac{(\sqrt{1+x}+\sqrt{1-x})}{(\sqrt{1+x}+\sqrt{1-x})} \\ &= \lim_{x\to 0} \frac{x(\sqrt{1+x}+\sqrt{1-x})}{(1+x-(1-x))} \\ &= \lim_{x\to 0} \frac{x(\sqrt{1+x}+\sqrt{1-x})}{2x} = \lim_{x\to 0} \frac{(\sqrt{1+x}+\sqrt{1-x})}{2} \\ &= \frac{(\sqrt{1}+\sqrt{1})}{2} = 1\end{aligned}$$

(c) $\lim_{x\to 4^-} \dfrac{2x+|2x|}{x+|x|}$.

Solution:

$$\lim_{x\to 4^-} \frac{2x+|2x|}{x+|x|} = \frac{2\cdot 4+|2\cdot 4|}{4+|4|} = \frac{16}{8} = 2$$

(d) $\lim_{x\to -\infty} e^x \cos(x + \frac{1}{x})$

Solution: Since for all $x \in \mathbb{R}$,

$$-1 \le \cos x \le 1,$$

then for all $x \neq 0$:

$$-1 \le \cos(x + \frac{1}{x}) \le 1$$

$$-e^x \le e^x \cos(x + \frac{1}{x}) \le e^x.$$

$$\text{Since } \lim_{x\to -\infty} -e^x = 0 \quad \text{and} \quad \lim_{x\to -\infty} e^x = 0,$$

$$\Rightarrow \quad \lim_{x\to \infty} e^x \cos(x + \frac{1}{x}) = 0 \text{ by the squeeze theorem.}$$

(e) $\lim_{x\to 1} \frac{\tan(x-1)}{2x-2}$

Solution:

$$\begin{aligned} \lim_{x\to 1} \frac{\tan(x-1)}{2x-2} &= \lim_{x\to 1} \frac{\sin(x-1)}{\cos(x-1)(2x-2)} \\ &= \lim_{x\to 1} \frac{\sin(x-1)}{(x-1)} \cdot \frac{1}{2\cos(x-1)} = \frac{1}{2} \end{aligned}$$

4. [**5 marks**] Let $f(x) = \dfrac{(x^2-9)(x^2-4)}{(x^4-x^3-6x^2)}$. Find all the horizontal and vertical asymptotes of f. Justify your answer by using appropriate limits.

Solution:

For horizontal asymptotes:

$$\begin{aligned}\lim_{x\to\infty} \frac{(x^2-9)(x^2-4)}{(x^4-x^3-6x^2)} &= \lim_{x\to\infty} \frac{(x^4-13x^2+36)}{(x^4-x^3-6x^2)} = \lim_{x\to\infty} \frac{\frac{x^4}{x^4}-\frac{13x^2}{x^4}+\frac{36}{x^4}}{\frac{x^4}{x^4}-\frac{x^3}{x^4}-\frac{6x^2}{x^4}} \\ &= \lim_{x\to\infty} \frac{1-\frac{13}{x^2}+\frac{36}{x^4}}{1-\frac{1}{x}-\frac{6}{x^2}} \\ &= 1\end{aligned}$$

$$\begin{aligned}\lim_{x\to-\infty} \frac{(x^2-9)(x^2-4)}{(x^4-x^3-6x^2)} &= \lim_{x\to-\infty} \frac{(x^4-13x^2+36)}{(x^4-x^3-6x^2)} = \lim_{x\to-\infty} \frac{\frac{x^4}{x^4}-\frac{13x^2}{x^4}+\frac{36}{x^4}}{\frac{x^4}{x^4}-\frac{x^3}{x^4}-\frac{6x^2}{x^4}} \\ &= \lim_{x\to\infty} \frac{1-\frac{13}{x^2}+\frac{36}{x^4}}{1-\frac{1}{x}-\frac{6}{x^2}} \\ &= 1\end{aligned}$$

Thus there is one horizontal asymptote, $y = 1$.

For vertical asymptotes we first simplify by factoring the function as much as possible:

$$f(x) = \frac{(x^2-9)(x^2-4)}{(x^4-x^3-6x^2)} = \frac{(x-3)(x+3)(x-2)(x+2)}{x^2(x^2-x-6)} = \frac{(x-3)(x+3)(x-2)(x+2)}{x^2(x-3)(x+2)}$$

So

$$f(x) = \frac{(x+3)(x-2)}{x^2} \text{ for all } x \in \mathbb{R},\ x \notin \{-2, 0, 3\}$$

Thus:

$$\lim_{x\to-2} \frac{(x+3)(x-2)}{x^2} = \frac{-4}{4} = -1$$

$$\lim_{x\to3} \frac{(x+3)(x-2)}{x^2} = \frac{6}{9} = \frac{2}{3}$$

$$\lim_{x\to0} f(x) = \lim_{x\to0} \frac{(x+3)(x-2)}{x^2} = -\infty$$

(as $x \to 0$ $(x+3)(x-2) \to 3(-2) = -6 < 0$ the denominator $x^2 \to 0^+$.)

This shows that f has one vertical asymptote at $x = 0$.

5. [**5 marks**] Given some numbers a and b, define $f : \mathbf{R} \to \mathbf{R}$ by

$$f(x) = \begin{cases} \frac{\sin(17x)}{\sin(x)} & \text{if } x < 0\,, \\ ax + b & \text{if } 0 \le x \le 1\,, \\ \ln(x) & \text{if } x > 1\,, \end{cases}$$

Find a and b so that f is continuous on $(-\pi/2, \infty)$. Fully justify your answers.

Solution:

Since $\sin 17x, \sin x$ are both continuous for all $x \in \mathbb{R}$ and $\sin x = 0$ for $x = k\pi$ for $k \in \mathbb{Z}$. Then $\dfrac{\sin(17x)}{\sin(x)}$ is continuous on $\left(-\frac{\pi}{2}, 0\right)$.

Since $ax + b$ is a polynomial then it is continuous on $(0,\ 1)$.

Also, $\ln x$ is continuous on $(0, \infty)$ therefore it is continuous on $(1, \infty)$.

For f to be continuous at $x = 0$ we need

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0).$$

So

$$\begin{aligned} \lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-} \frac{\sin(17x)}{\sin(x)} \\ &= \lim_{x\to 0^-} \frac{\sin(17x)}{17x} \frac{x}{\sin x} \frac{17x}{x} \\ &= \lim_{x\to 0^-} \frac{\sin(17x)}{17x} \cdot \lim_{x\to 0^-} \frac{x}{\sin(x)} \cdot \lim_{x\to 0^-} \frac{17)}{1} \\ &= 1 \cdot 1 \cdot 17 \qquad \because \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1 \\ &= 17 \end{aligned}$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} ax + b = b = f(0)$$

For continuity at 0 we need $b = 17$.

For continuity at 1 we need

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1).$$

$$\begin{aligned} \lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-} ax + b = a + b = a + 17 = f(1) \\ \lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+} \ln(x) = \ln(1) = 0 \end{aligned}$$

For continuity at 1 we need $a + 17 = 0$ or $a = -17$.

Thus, if $a = -17$ and $b = 17$, then f is continuous on $(\pi/2, \infty)$.

6. [**10 marks**] Determine if there is a solution to

$$x^{5/3} + x^{2/3} + 6 = 0\,.$$

Justify your answer and give the full statement of the theorem you used.

Solution: Since the function $f(x) = x^{5/3}+x^{2/3}+6$ is the sum of powers of root functions that are continuous on $\mathbb{R}$, then f is continuous on $\mathbb{R}$.

$f(0) = 6 > 0$ and

$$\begin{aligned} f(-8) &= (-8)^{5/3} + (-8)^{2/3} + 6 \\ &= (\sqrt[3]{-8})^5 + (\sqrt[3]{-8})^2 + 6 \\ &= (-2)^5 + (-2)^2 + 6 \\ &= -32 + 4 + 6 = -22 < 0 \end{aligned}$$

We have that $f(-8) < 0 < f(0)$ and since f is continuous on $[-8, 0]$, then by the Intermediate Value Theorem there is some $c \in (-8, 0)$ such that $f(c) = 0$.

Thus there is at least one solution to $x^{5/3} + x^{2/3} + 6 = 0$ in $(-8, 0)$.

The Intermediate Value Theorem: If f is continuous on $[a, b]$ and w is between $f(a)$ and $f(b)$, then there is some $c \in (a, b)$ such that $f(c) = w$.

7. (a) [**5 marks**] Let $f(x) = \dfrac{1}{x-2}$. Use **the definition** of the derivative to find $f'(x)$.

Solution:

$$\begin{aligned}
\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{\frac{1}{x+h-2}-\frac{1}{x-2}}{h} \\
&= \lim_{h\to 0}\frac{\frac{1}{x+h-2}-\frac{1}{x-2}}{h} \\
&= \lim_{h\to 0}\frac{\frac{(x-2)-(x+h-2)}{(x-2)(x+h-2)}}{h} \\
&= \lim_{h\to 0}\frac{-h}{(x-2)(x+h-2)h} \\
&= \lim_{h\to 0}\frac{-1}{(x-2)(x+h-2)} \\
&= \frac{-1}{(x-2)^2}
\end{aligned}$$

(b) [**5 marks**] Find the line tangent line to the graph of $f(x) = \dfrac{1}{x-2}$ at the point $(4, 1/2)$.

Solution:

Since $f(4) = \dfrac{1}{4-2} = \dfrac{1}{2}$, then the point $(4, 1/2)$ is on the graph of f.

The slope of the tangent line at $(4, 1/2)$ is $m = f'(4) = \dfrac{-1}{(4-2)^2} = \dfrac{-1}{4}$.

Thus:

$$\frac{-1}{4} = \frac{y-\frac{1}{2}}{x-4}$$

Thus, the equation of the tangent line is $y = \dfrac{1}{2} - \dfrac{1}{4}(x-4) = -\dfrac{1}{4}x + \dfrac{3}{2}$.

8. [**5 marks each; 25 marks**] For the following given functions use differentiation rules to find their required derivatives. Do not simplify.

(a) [**5 marks**] $f(x) = e^x(1 + \sec x)$. Find $f'(x)$.

Solution: Using the product rule and the chain rule:

$$f'(x) = e^x(1 + \sec x) + e^x \sec x \tan x$$

(b) [**5 marks**] $g(x) = \dfrac{\arcsin(x)}{\sqrt{1-x^2}}$. Find $g'(x)$.

Solution:

$$\begin{aligned} g'(x) &= \frac{\frac{1}{\sqrt{1-x^2}} \cdot \sqrt{1-x^2} - \arcsin x \cdot \frac{1}{2\sqrt{1-x^2}} \cdot (-2x)}{(\sqrt{1-x^2})^2} \\ &= \frac{\frac{\sqrt{1-x^2}}{\sqrt{1-x^2}} + \frac{x \arcsin x}{\sqrt{1-x^2}}}{1-x^2} \\ &= \frac{\frac{\sqrt{1-x^2}+x\arcsin x}{\sqrt{1-x^2}}}{1-x^2} \\ &= \frac{\sqrt{1-x^2} + x \arcsin x}{\sqrt{1-x^2}(1-x^2)} = \frac{\sqrt{1-x^2} + x \arcsin x}{(1-x^2)^{3/2}} \end{aligned}$$

(c) [**5 marks**] $g(x) = 2\sin(\ln(x))\cos(\ln(x))$. Find $g(x)$.

Solution: Using the product rule and the chain rule:

$$g'(x) = 2\cos(\ln(x))\left(\frac{1}{x}\right)\cos(\ln(x)) + 2\sin(\ln(x))(-\sin(\ln x))\left(\frac{1}{x}\right).$$

$$g'(x) = \left(\frac{2}{x}\right)[\cos^2(\ln(x)) - \sin^2(\ln(x))] = \left(\frac{2}{x}\right)\cos(2\ln x).$$

Or using the double angle identity $\sin 2y = 2\sin y \cos y$ we get that $g(x) = 2\sin(\ln(x))\cos(\ln(x)) = \sin(2\ln x)$, thus

$$g'(x) = \cos(2\ln x) \cdot \frac{2}{x}.$$

(d) [**5 marks**] $h(x) = (x+1)^{\tan^2 x}$. Find $h(x)$.

Solution: Logarithmic differentiation is needed.
Let $y = h(x)$. Then $y = (x+1)^{\tan^2 x}$.
Taking the natural logarithm of both sides and simplifying:

$$\ln y = (\tan^2 x)\ln(x+1)$$

Using implicit differentiation:

$$\frac{1}{y} \cdot y' = 2(\tan x)(\sec^2 x)\ln(x+1) + \tan^2 x \frac{1}{(x+1)}$$

Thus:

$$\begin{aligned} y &= y\left(2(\tan x)(\sec^2 x)\ln(x+1) + \frac{\tan^2 x}{(x+1)}\right) \\ &= (x+1)^{\tan^2 x}\left(2(\tan x)(\sec^2 x)\ln(x+1) + \frac{\tan^2 x}{(x+1)}\right) \end{aligned}$$

(e) [**5 marks**] If $e^{xy} = e^{4x} - e^{5y}$, find $\frac{dy}{dx}$.

Solution: Using implicit differentiation on $e^{xy} = e^{4x} - e^{5y}$, we get:

$$e^{xy}(y + xy') = e^{4x}4 - e^{5y}5y'.$$

Isolating all terms with y' on one side of the equation we get:

$$ye^{xy} + xy'e^{xy} = e^{4x}4 - e^{5y}5y'.$$

$$xy'e^{xy} + e^{5y}5y' = 4e^{4x} - ye^{xy}$$
$$y'(xe^{xy} + 5e^{5y}) = 4e^{4x} - ye^{xy}$$

Finally:

$$y' = \frac{4e^{4x} - ye^{xy}}{xe^{xy} + 5e^{5y}}$$

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