# University of Toronto at Scarborough
# Department of Computer & Mathematical Sciences

**Term Test**

**MATB24** Linear Algebra II

Examiner: X. Jiang

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Duration: 110 minutes

**1.** **(15 points)**

. a) Give the definition of a general vector space.

A <u>General vector space</u> is a set V of objects called vectors over a field F, together with a rule of adding two vectors v and w to produce a vector $\bar{u} \oplus \bar{v}$ in V and a rule for multiplying any vector v in V by any r in F to produce a vector $r \otimes \bar{v}$ in V. Moreover, for all the vectors in V, the following properties are satisfied:

$\forall \bar{u}, \bar{v}, \bar{w} \in V$, $\forall$ r, s $\in$ F,

Properties of vector addition:

A1: $(\bar{u} \oplus \bar{v}) \oplus \bar{w} = \bar{u} \oplus (\bar{v} \oplus \bar{w})$ an associatelaw

A2: $\bar{u} \oplus \bar{w} = \bar{w} \oplus \bar{u}$ a commutative law

A3: $\bar{0} \oplus \bar{v} = \bar{v}$ $\bar{0}$ as additive identity

A4: $\bar{v} \oplus (-\bar{v}) = 0$ $-\bar{v}$ as additive inverse of $\bar{v}$

Properties of vector scalar multiplication:

S1: $r \otimes (\bar{v} \oplus \bar{w}) = (r \otimes \bar{v}) \oplus (r \otimes \bar{w})$ a distributive law

S2: $(r \oplus s) \otimes \bar{v} = (r \otimes \bar{v}) \oplus (s \otimes \bar{v})$ a distributive law

S3: $r \otimes (s \otimes \bar{v}) = (rs) \otimes \bar{v}$ a associative law

S4: $1 \otimes \bar{v} = \bar{1}$ preservation of scale

b) b) Let $f(x) = x^3$ defined on R and Let $V = \{f(x + t) \mid t \in R\}$ . Define $\oplus$ and $\otimes$ by

$f(x + s) \oplus f(x + t) = f(x + s + t)$ and $c \otimes f(x + t) = f(x + ct)$.

1) Determine whether or not V is a real vector space.

**Your answer:**____Yes____________

2) If your answer in 1) is “yes”, describe the “zero vector” in V and “the additive inverse” of $\bar{v} \in V$. If your answer in 1) is :No”, which axiom(s) in (a) fails? Give an example.

$\bar{0} = f(x + 0)$.

$\forall \bar{v} = f(x + t)$ in V, $-\bar{v} = f(x - t)$.

**2. (10 points)** Prove that {1, sinx, sin2x, sin3x} is linearly independent.

Proof: Let $r_1 + r_2\sin x + r_3\sin 2x + r_4\sin 3x = 0$, $r_i \in R$.

Set x =0, we have $r_1 = 0$ (1)

Set $x = \pi/2$, we have $r_1 + r_2 - r_4 = 0$. (2)

Set $x = \pi/3$, we have $r_1 + \frac{1}{\sqrt{2}} r_2 + r_3 + \frac{1}{\sqrt{2}} r_4 = 0$. (3)

Set $x = \pi/4$, we have $r_1 + \frac{1}{2} r_2 + \frac{1}{2} r_3 = 0$. (4)

Solve equations (1) – (4) and obtain $r_1 = r_2 = r_3 = r_4 = 0$.

Therefore, {1, sinx, sin2x, sin3x} is linearly independent.

## 3. (20 points) True or False:

1) False 2) True 3)True 4) False 5) True 6) True 7) True 8) True 9) True 10) False

## 4. (12 points)

1) Let T: $P_3 \to M_{22}$ be the linear transformation given by $T(ax^3 + bx^2 + cx + d) = \begin{bmatrix} a-b & b+c \\ c+d & d-a \end{bmatrix}$.

Let B = $\{1, x, x^2, x^3\}$ be a basis for $P_3$ and $B' = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

Find the matrix representation relative to B, B′.

$$A = \begin{bmatrix} 0 & 0 & -1 & 1 \\ -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & -2 \end{bmatrix}$$

2) Computer $T(x^3 + x^2 - x + 1)_{B'}$.

$T(x^3 + x^2 - x + 1)_{B'} = [0, 0, 0, 0]$

## 5. (14 points)

1) Suppose $V_1$ and $V_2$ are subspaces of a vector space V. Define

$$W = \{ \bar{u} + \bar{v} \mid \bar{u} \in V_1,\ \bar{v} \in V_2 \}.$$

Prove that W is a subspace of V.

Proof:

a) Since $V_1$ and $V_2$ are subspaces of a vector space V , $\bar{0} \in V_1$ and $\bar{0} \in V_2$,.

$\bar{0} = \bar{0} + \bar{0} \in W$.

b) $\forall \bar{w}_1, \bar{w}_2 \in W$, there are $\bar{u}_1, \bar{u}_2 \in V_1$, $\bar{v}_1, \bar{v}_2 \in V_2$ such that

$$\bar{w}_1 = \bar{u}_1 + \bar{v}_1, \bar{w}_2 = \bar{u}_2 + \bar{v}_2$$

Since $V_1$ and $V_2$ are subspaces of a vector space V,

$$\bar{u}_1 + \bar{u}_2 \in V_1, \bar{v}_1 + \bar{v}_2 \in V_2.$$

$$\bar{w}_1 + \bar{w}_2 = (\bar{u}_1 + \bar{v}_1) + (\bar{u}_2 + \bar{v}_2) = (\bar{u}_1 + \bar{u}_2) + (\bar{v}_1 + \bar{v}_2) \in W.$$

c) $\forall \bar{w} = \bar{u} + \bar{v} \in W$,. $\forall r \in R$,

Since $V_1$ and $V_2$ are subspaces of a vector space V, $r\bar{u} \in V_1$ and $r\bar{v} \in V_2$.

Then $r\bar{w} = r\bar{u} + r\bar{v} \in W$.

Therefore, $W$ is a subspace of $V$.

2) Let V and V′ be vector spaces having the same finite dimension, and let T: V → V′ be a linear transformation. Prove that T is one-to-one if and only if range(T) = V′.

Proof:

Since dim(V) = dim(V′),

by dimension theorem we have

$$\dim(\text{range}(T)) + \dim(\ker(T)) = \dim(V) = \dim(V').$$

Then

T is one-to-one $\Leftrightarrow$ ker(T) = { $\bar{0}$ } $\Leftrightarrow$ dim(ker(T) = 0 $\Leftrightarrow$ dim(range(T))=dim(V′)

$\Updownarrow$

T is onto $\Leftrightarrow$ range(T) = V′

**6. (16 points)** Let T: $P_3 \to P_2$ given by T(p(x)) = p′′(x) + p′(x) + p(0).

1) Show that T is a linear transformation.

$\forall$ p(x), q(x) $\in P_3$, $\forall$ a, b $\in$ R,

T(ap(x) + bq(x)) = (ap(x) +bq(x))′′ + (ap(x) + bq(x))′+ ap(0) +bq(0)

= a(p′′(x) + p′(x) + p(0)) + b(q′′(x) + q′(x) + q(0))

= aT(p(x)) + bT(q(x))

2) Find bases for the ker(T) and range(T).

Ker(T) = {p(x) | p′′(x) + p′(x) + p(0) = 0, p(x) $\in P_3$}

= $\{ax^3 + bx^2 + cx + d \mid 6ax + 2b + 3ax^2 + 2bx + c + d = 0,\ a, b, c, d \in R\}$

= $\{ax^3 + bx^2 + cx + d \mid a = 0,\ 6a + 2b = 0,\ 2b + c + d = 0,\ a, b, c, d \in R\}$

= {cx – c | c $\in$ R} = sp(x -1)

A basis for ker(T) is {x – 1}.

range(T) = { p′′(x) + p′(x) + p(0) | p(x) $\in P_3$}

= $\{3ax^2 + (6a + 2b)x + 2b + c + d = 0,\ a, b, c, d \in R\}$

= $\{a(3x^2 + 6x) + b(2x + 2) + c + d \mid a, b, c, d \in R\}$

= $sp(3x^2 + 6x,\ 2x + 2,\ 1)$.

A basis for range(T) is $\{ 3x^2 + 6x,\ 2x + 2,\ 1\}$

3) Determine whether T is one-to-one and/or onto.

Since ker(T) = sp(x -1) $\neq \{0\}$, so T is not one-to-one.

dim(range(T)) = 3 = dim($P_2$) , so T is not onto.

**7. (13 points)**

1) Show that the following identity holds for vectors $\bar{u}$, $\bar{v}$ in any inner product space.

$$<\bar{u},\ \bar{v}> = \frac{1}{4}\|\bar{u}+\bar{v}\|^2 - \frac{1}{4}\|\bar{u}-\bar{v}\|^2$$

Proof: $\|\bar{u}+\bar{v}\|^2 = <\bar{u}+\bar{v},\ \bar{u}+\bar{v}> = <\bar{u},\ \bar{u}> + 2<\bar{u},\ \bar{v}> + <\bar{v},\ \bar{v}>$ (1)

$\|\bar{u}-\bar{v}\|^2 = <\bar{u}-\bar{v},\ \bar{u}-\bar{v}> = <\bar{u},\ \bar{u}> - 2<\bar{u},\ \bar{v}> + <\bar{v},\ \bar{v}>$ (2)

Eqn.(1) – (2): $\|\bar{u}+\bar{v}\|^2 - \|\bar{u}-\bar{v}\|^2 = 4<\bar{u},\ \bar{v}>$ (3)

Both sides of eqn.(3) dividing 4 will give the result.

2) Let C[0, 1] denote the vector space of continuous functions from [0, 1] to R with an inner Product defined by

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx.$$

a) Find the magnitude of the polynomial p(x) = x + 1.

$$\|x+1\|^2 = < x+1, \quad x+1 > = \int_0^1 (x+1)(x+1)dx = \frac{7}{3}$$

Therefore, $\|x+1\| = \sqrt{\frac{7}{3}}$.

b) Find the distance and angle between $x^2$ to x.

$$d^2(x^2, \quad x) = \|x^2 - x\|^2 = < x^2 - x, \quad x^2 - x > = \int_0^1 (x^2 - x)(x^2 - x)dx = \frac{1}{30}$$

Therefore, $d(x^2, \quad x) = \sqrt{\frac{1}{30}}$

$$\cos\theta = \frac{< x^2, \quad x >}{\|x^2\| \quad \|x\|} = \frac{\int_0^1 x^3 dx}{\sqrt{\int_0^1 x^4 dx}\sqrt{\int_0^1 x^2 dx}} = \frac{\frac{1}{4}}{\sqrt{\frac{1}{5}}\sqrt{\frac{1}{3}}} = \frac{\sqrt{15}}{4}$$

Then, $\theta = \cos^{-1}\frac{\sqrt{15}}{4}$.