Linear Algebra Midterm 1 Review

Definitions and Concepts (you should be able to state/apply the definition/concept):

Elementary row operations:
1. (Replacement) Replace one row by itself plus a multiple of another.
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply all entries in a row by a nonzero constant.

Elementary row operations performed on the *augmented matrix* preserve the solution set.

Span$\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$:

• If $v_1, \ldots, v_p$ are vectors in $\mathbb{R}^n$, then the set of all linear combinations of $v_1, \ldots, v_p$ is denoted Span$\{v_1, \ldots, v_p\}$. That is, Span$\{v_1, \ldots, v_p\}$ is a collection of all vectors that can be written in the form $c_1v_1 + c_2v_2 + \ldots + c_pv_p$ with $c_1, \ldots, c_p$ scalars.

Product of a matrix and a vector:

• If A is an $m \times n$ matrix with columns $a_1, \ldots, a_n$ and if x in $\mathbb{R}^n$, then the product of A and x, denoted by Ax, is a linear combination of the columns of A using the corresponding entries in x as coefficients; that is

$$Ax = [a_1\, a_2\, \ldots a_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1a_1 + x_2a_2 + \ldots + x_na_n$$

Systems of linear equations, vector equations, and matrix equations

• If A is an $m \times n$ matrix with columns $a_1, \ldots, a_n$ and if b is in $\mathbb{R}^m$, the matrix equation $Ax = b$ has the same solution set as the vector equation $x_1a_1 + x_2a_2 + \ldots + x_na_n = b$ which in turn has the same solution set as the system of linear equations whose augmented matrix is $[a_1\, a_2\, \ldots\, a_n\, b]$.

Linear Independence and Dependence

• An indexed set of vectors $\{v_1, \ldots, v_p\}$ in $\mathbb{R}^n$ is said to be linearly *independent* if the vector equation $x_1v_1 + x_2v_2 + \ldots + x_pv_p = 0$ has only the trivial solution, i.e. $x_1 = x_2 = \ldots = x_p = 0$.

• The set $\{v_1, \ldots, v_p\}$ is said to be linearly *dependent* if there exist scalars $c_1, \ldots, c_p$ not all zero such that $c_1v_1 + c_2v_2 + \ldots + c_pv_p = 0$.

• For vectors $v_1, \ldots, v_p$ in $\mathbb{R}^n$ checking whether the set $\{v_1, \ldots, v_p\}$ is linearly *independent* is the same as checking whether the system $Ax = 0$ has nontrivial solutions where $A = [v_1\, \ldots\, v_p]$.

Linear transformations

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. The set $\mathbb{R}^n$ is said to the *domain* of T, and $\mathbb{R}^m$ is said to be the *codomain* of T. The *range* of T is the set of all values of T, that is, it consists of $T(x)$ where x is in $\mathbb{R}^n$. In general, the range is not the same as the codomain of T.

• A transformation (or mapping) $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ is linear if:

1. $T(u+v) = T(u) + T(v)$ for all u, v in the domain of T
2. $T(cu) = cT(u)$ for all u and all scalars c.

• A transformation $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ defined by multiplication by an $m \times n$ matrix A, $T(x) = Ax$, is linear.

Useful Theorems:

Existence and Uniqueness Theorem: A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column if and only if an echelon form of the augmented matrix has no row of the form $[0\,0\,0\,\ldots\,0\,d]$ with nonzero d. If a linear system is consistent, then the solution set contains either
(i) a unique solution when there are no free variables, or
(ii) infinitely many solutions when there is at least one free variable.

Theorem: Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or there are all false.
(a) For each b in $\mathbb{R}^m$, the equation $Ax = b$ has a solution.
(b) Each b in $\mathbb{R}^m$ is a linear combination of columns of A.
(c) The columns of A span $\mathbb{R}^m$.
(d) A has a pivot in every row.

Theorem: Suppose the equation $Ax = b$ is consistent for some b, and let p be a solution. Then the solution set of $Ax = b$ is the set of all vectors of the form $w = p + v_h$ where v_h is any solution of the homogeneous equation $Ax = 0$.

Fact: A homogeneous equation $Ax = 0$ has infinitely many solutions if and only if there is at least one free variable.

Theorem: Any set $\{v_1, \ldots, v_p\}$ in $\mathbf{R}^n$ is linearly dependent if $p > n$.

Practice Problems

Section 1.4: 22, 23, 24, 29, 30, 32;

Section 1.5: 7-12, 12, 23, 24, 28-31;

Section 1.7: 7-10, 21, 22, 31, 33, 34, 38;

Section 1.8: 2, 3, 11, 21, 23, 32.