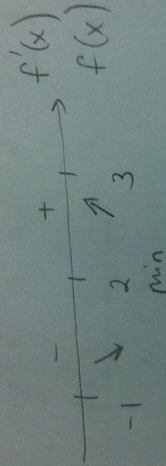


1. (a) (5 points) Consider the function

$$f(x) = e^{(x-2)^2}$$

Find the critical points of $f(x)$, and determine the absolute maximum and minimum values of $f(x)$ on the interval $[-1, 3]$.

$$f'(x) = e^{(x-2)^2} \cdot 2(x-2)$$



$$\text{absolute min } f(x) = f(2) = 1$$

$$\left. \begin{array}{l} f(-1) > f(3) \\ e^9 > e^1 \end{array} \right\} \Rightarrow f(-1) \text{ is the absolute max.}$$

$$f(-1) = e^9$$

critical points: $x = 2$

4
(b) (5 points) Consider the curve

$$x^5 y + 2xy^2 = 10$$

Determine the equation of the tangent line to the curve at the point $(1, 2)$.

$$(x^5 y + 2xy^2)' = (10)'$$

$$5x^4 y + y' \cdot x^5 + 2y^2 + 2x \cdot 2y \cdot y' = 0$$

$$y' = -\frac{5x^4 y + 2y^2}{x^5 + 4xy}$$

$$y'(1, 2) = -\frac{5 \cdot 2 + 2 \cdot 4}{1 + 8} = -2 = k$$

$$y_{\text{tan}} = -2x + b$$

$$-2 + b = 2$$

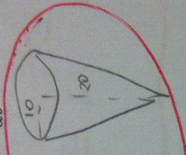
$$b = 4$$

$$y_{\text{tan}} = -2x + 4 \quad \checkmark$$

2. Beer flows out the bottom of a funnel (a right circular cone with tip pointing downwards) with radius 10 cm and height 20 cm. You measure the volume and depth of the beer in the funnel as it flows. Let V be the volume of beer in the funnel and h its depth at time t .

- (a) (3 points) Express V as a function of h , then use this to express $\frac{dV}{dt}$ in terms of h and $\frac{dh}{dt}$.

3



$$2r = h$$

$$V = \frac{1}{3} \pi r^2 \cdot h = \frac{1}{3} \pi \frac{h^2}{4} \cdot h = \frac{\pi h^3}{12}$$

$$V' = \frac{\pi}{12} 3h^2 h' = \frac{\pi h^2 h'}{4}$$

- (b) (2 points) For some constant C it is known that the beer flow satisfies

$$\frac{dV}{dt} = C\sqrt{h} \quad (\text{Torricelli's Law})$$

Is C positive or negative (explain)? If h is measured in centimeters and t is measured in seconds, what are the units of C ?

2

$$\frac{\pi h^2 h'}{4} = C\sqrt{h} \quad ; \quad C = \frac{\pi h^2 h'}{4\sqrt{h}} \quad \left(\frac{\pi h^2}{4\sqrt{h}} > 0 ; \right.$$

h decreases $\rightarrow h'$ negative $\rightarrow C' \rightarrow$ negative)

(beer flows out)

V' is measured in cm^3/sec

$\sqrt{h}$ is $\sqrt{\text{cm}}$
is measured in $\text{cm}^{1/2}$

C is $\text{cm}^{5/2}/\text{sec}$

(c) (3 points) Show that

$$\frac{dh}{dt} = \frac{4C}{\pi} h^{-3/2}.$$

$$\frac{\pi h^2 \cdot h'}{4} = V' = C\sqrt{h}$$

$$h' = \frac{4C\sqrt{h}}{\pi h^2} = \frac{4C}{\pi} h^{-3/2}$$

(d) (1 point) In an experiment to determine the value of C , you measure the beer level in the funnel to be falling at a rate of 5 cm/s when $h = 4$ cm. Determine the value of C .

$$C = \frac{\pi h^2 h'}{4\sqrt{h}}$$

$$C = \frac{\pi \cdot 16 \cdot (-5)}{4 \cdot 2} = -10\pi$$

3. The price p (in dollars) for a bag of dog treats and the hourly number of sales q are related by $q = \sqrt{45 - 10p}$. Currently, you charge \$2 per bag.

(a) (3 points) Compute the elasticity of demand ϵ for the cat treats at this price.

$$q = \sqrt{45 - 10p} \quad q(2) = 5$$

$$q' = \frac{-10}{2\sqrt{45-10p}} = -\frac{5}{\sqrt{45-10p}} \quad \textcircled{1} \quad q'(2) = -\frac{5}{5} = -1$$

$$\epsilon = \frac{\Delta q}{\Delta p} \cdot \frac{p}{q} = q' \cdot \frac{p}{q} = -1 \cdot \frac{2}{5} = -0.4$$

- (b) (2 points) Suppose the price is raised from \$2 and this yields a 3% decrease in demand. Estimate the percentage by which the price was changed.

$$\frac{\Delta q}{q} = -0.03$$

$$\frac{\Delta p}{p} = \frac{\Delta q}{q} \cdot \epsilon = (-0.03) \cdot \left(-\frac{5}{2}\right) = \frac{3 \cdot 5}{100 \cdot 2} = \frac{15}{200} = 7.5\%$$

price increased by 7.5%

2

Now suppose $\epsilon = -1/104$ at a certain price for the cat treats. For part (iv) include an explanation or no credit will be given.

- i. (1 point) State whether the cat treats are price elastic, price inelastic or price unit elastic at this price.

Answer:

price inelastic

- ii. (1 point) If this price is raised slightly, state whether the revenue will increase, decrease, stay constant, or whether there is insufficient information to determine this.

Answer:

Revenue will increase

- iii. (1 point) If this price is raised slightly, state whether the demand will increase, decrease, stay constant, or whether there is insufficient information to determine this.

Answer:

demand will decrease

- iv. (2 points) If this price is raised slightly, state whether the profit will increase, decrease, stay constant, or whether there is insufficient information to determine this. Assume here that the cost function $C(q)$ is increasing.

Answer:

Profit will increase

provide an explanation in the space below

$$P = R - C$$

$$P' = R' - C'$$

$R(q)$ in R' is positive

$C(q)$ dec $\rightarrow C'$ is negative

P' positive!

$\Rightarrow$ not enough info to say whether

P' will be positive

or negative $\Rightarrow$

unknown whether profit's

4. In the year 2000, a student takes out a \$30 000 loan at an interest rate of r (interest compounded continuously). The student makes one loan payment of \$60 000 in 2015, then one more loan payment of \$ L in 2020 after which the student owes nothing.

- (a) (3 points) If the student's debt is \$60 000 in 2005, what is r ?

$$60000 = 30000 \cdot e^{rt}$$

$$e^{rt} = 2$$

$$r = \frac{\ln 2}{5}$$

t - time in years

- (b) (3 points) At what rate (in dollars per year) is the student's debt increasing in 2010? (use the value of r from part (a))

$$A'(t) = (30000 \cdot e^{\frac{\ln 2}{5} \cdot t})' = 30000 \cdot e^{\frac{\ln 2}{5} \cdot t} \cdot \frac{\ln 2}{5}$$

where A is student's debt

$$A'(10) = 6000 \cdot \ln 2 \cdot 4 = 24000 \cdot \ln 2$$

- (c) (4 points) What is L ? (use the value of r from part (a))

$$L = (30000 \cdot e^{\frac{\ln 2}{5} \cdot 20} - 60000) \cdot e^{\frac{5 \cdot \ln 2}{5}}$$

$$= 30000 (e^{3 \ln 2} - 2) \cdot e^{\ln 2} = 30000 (2^3 - 2) \cdot 2$$

$$= 360000 \$$$

$$30000 \cdot e^{\frac{\ln 2}{5} \cdot t} = 30000 \cdot 2$$

$$0.62 = x$$

$$x = \log_e 2$$

$$x = 2$$

5. Consider the following function

$$f(x) = \frac{x^2 - 9}{x^2 - 4}$$

You may assume that

$$f'(x) = \frac{10x}{(x^2 - 4)^2}; \quad f''(x) = -\frac{10(3x^2 + 4)}{(x^2 - 4)^3}$$

(a) (1 point) Find the x - and y -intercepts of the graph of the function.

$$x \text{ intercepts: } (0, \frac{9}{4}); (3, 0); (-3, 0)$$

(b) (1 point) Find the domain of $f(x)$

$$D = \mathbb{R} \setminus \{\pm 2\}$$

↓
excluding

(c) (3 points) Find all critical points, and intervals of increase and decrease for $f(x)$.

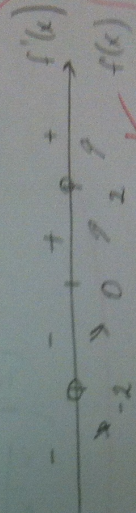
$$f'(x) = \frac{10x}{(x^2 - 4)^2}$$

Critical points: ✓

$$x = -2; 0; 2$$

intervals of increase: $(0, 2) \cup (2, \infty)$

intervals of decrease: $(-\infty, -2) \cup (-2, 0)$



$$f''(x) = -\frac{10(3x^2 + 4)}{(x^2 - 4)^3}$$