25 marks.

Math 2015 Applied Multivariate and Vector Calculus: Test2

Friday October 9, 2009
10:30am to 11:20am

Name: SOLUTIONS **Student Number:**

Instructions: Complete all 5 of the following problems in the space provided. Notes and calculators are not permitted. All cell phones and pagers are to be turned off.

1. The radius of a right circular cone is increasing at a rate of 1/2 in/s while its height is decreasing at a rate of 2 in/s. At what rate is the volume of the cone changing when the radius is 12 in and the height is 10 in? (Hint: the volume of a cone is $V = \pi r^2 h/3$).

5 marks.

$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{dV}{dt} = \frac{\pi}{3}\frac{d}{dt}(r^2 h) = \frac{\pi}{3}\left[2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right] \quad \text{— (2)}$$

Given $\frac{dr}{dt} = +\frac{1}{2}\,\frac{\text{in}}{\text{s}}$ $\quad \frac{dh}{dt} = -2\,\frac{\text{in}}{\text{s}}$ — (1)

$$\frac{dV}{dt} = \frac{\pi}{3}\left[2(12\text{ in})(10\text{ in})\left(\frac{1}{2}\frac{\text{in}}{\text{s}}\right) + (12\text{ in}^2)\left(-2\frac{\text{in}}{\text{s}}\right)\right] \quad \text{— (1)}$$

$$\left.\begin{aligned} &= \frac{\pi}{3}(-168\text{ in}^3/\text{s}) \\ &= -56\pi\text{ in}^3/\text{s} \end{aligned}\right\} (1)$$

2. Find the limit if it exists, or show that the limit does not exist for cases

5 marks.

$$\text{(a)} \lim_{(x,y)\to(0,0)} \frac{xy\cos y}{3x^2+y^2} \quad \text{and} \quad \text{(b)} \lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2}.$$

(a) Let $f(x,y) = \dfrac{xy\cos y}{3x^2+y^2}$

On the x-axis $f(x,0) = 0$ for $x \neq 0$ and $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along the x-axis. $\Big\}$ 1

Approaching $(0,0)$ along the line $y = x$

$f(x,x) = \frac{1}{4}\cos x$

So $f(x,x) \to \frac{1}{4}$ as $(x,x) \to (0,0)$ $\Big\}$ 1

Thus the limit does not exist. — 1

(b) Let $f(x,y) = \dfrac{x^4-y^4}{x^2+y^2}$

$$= \frac{(x^2-y^2)\cancel{(x^2+y^2)}}{\cancel{(x^2+y^2)}} = x^2-y^2 \quad \text{— 1}$$

Thus $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 0.$ — 1

3. Determine the tangent plane and normal line to the surface

5 marks.

$$x^3 + 3xyz + 2y^3 - z^3 = -15$$

at the point $(1, -1, 2)$.

Let $F(x,y,z) = x^3 + 3xyz + 2y^3 - z^3$

The tangent plane at point $\vec{r}_0 = (x_0, y_0, z_0)$ is given by

$$\nabla F(x_0, y_0, z_0) \cdot [\vec{r} - \vec{r}_0] = 0$$

where $\vec{r} = (x, y, z)$. Compute the gradient:

$$\nabla F = (3x^2 + 3yz)\hat{\imath} + (3xz + 6y^2)\hat{\jmath} + (3xy - 3z^2)\hat{k} \quad \text{— (1)}$$

Hence $\nabla F(1,-1,2) = (3-6)\hat{\imath} + (6+6)\hat{\jmath} + (-3-12)\hat{k}$

$$= -3\hat{\imath} + 12\hat{\jmath} - 15\hat{k} \quad \text{— (1)}$$

Since $(\vec{r} - \vec{r}_0) = (x-1, y+1, z-2)$ the tangent plane is given by

$$(-3, 12, -15) \cdot (x-1, y+1, z-2) = 0 \quad \text{— (1)}$$

or $$-3(x-1) + 12(y+1) - 15(z-2) = 0 \quad \text{— (1)}$$

Further, the normal line is easily written as

$$\frac{x-1}{-3} = \frac{y+1}{12} = \frac{z-2}{-15} \quad \text{— (1)}$$

4. The function $f(x, y)$ at the point $(1, 2)$ has a directional derivative equal to 2 in the direction towards $(2, 2)$ and equal to -2 in the direction towards $(1, 1)$. What is the directional derivative in the direction towards $(2, 3)$?

5 marks.

Let $\vec{u} = (2-1)\hat{\imath} + (2-2)\hat{\jmath} = \hat{\imath}$ So unit vector is

$$\hat{u} = \frac{\vec{u}}{\|\vec{u}\|} = \hat{\imath}$$

Also let $\vec{v} = (1-1)\hat{\imath} + (1-2)\hat{\jmath} = -\hat{\jmath}$, so unit vector is

$$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|} = -\hat{\jmath}$$

We are given: $D_{\hat{u}} f(1,2) = 2 = \hat{\imath} \cdot \nabla f(1,2)$ — 1

$D_{\hat{v}} f(1,2) = -2 = -\hat{\jmath} \cdot \nabla f(1,2)$ — 1

Defining $\vec{w} = (2-1)\hat{\imath} + (3-2)\hat{\jmath} = \hat{\imath} + \hat{\jmath}$, a unit vector is

$$\hat{w} = \frac{\vec{w}}{\|\vec{w}\|} = \frac{1}{\sqrt{2}}(\hat{\imath} + \hat{\jmath})$$ — 1

We wish to compute

$$D_{\hat{w}} f(1,2) = \hat{w} \cdot \nabla f(1,2) = \frac{1}{\sqrt{2}}(\hat{\imath} + \hat{\jmath}) \cdot \nabla f(1,2)$$ — 1

$$= \frac{1}{\sqrt{2}}\left[\hat{\imath} \cdot \nabla f(1,2) + \hat{\jmath} \cdot \nabla f(1,2)\right]$$

$$= \frac{1}{\sqrt{2}}[2 + 2] = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$ — 1

5. Consider the function $z = f(x,y)$ where $x = s+t$ and $y = s-t$. Show that

5 marks.

$$\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = \frac{\partial z}{\partial s}\frac{\partial z}{\partial t}$$

By the Chain Rule:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} \quad \text{— } 1$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} \quad \text{— } 1$$

Since $x = s+t$ and $y = s-t$

$$\left.\begin{aligned} \frac{\partial x}{\partial t} &= 1 \qquad \frac{\partial x}{\partial s} = 1 \\ \frac{\partial y}{\partial t} &= -1 \qquad \frac{\partial y}{\partial s} = 1 \end{aligned}\right\} 1$$

Hence

$$\left.\begin{aligned} \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \\ \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} \end{aligned}\right\} 1$$

Thus

$$\left.\begin{aligned} \frac{\partial z}{\partial s}\cdot\frac{\partial z}{\partial t} &= \left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right)\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right) \\ &= \left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 \end{aligned}\right\} 1$$