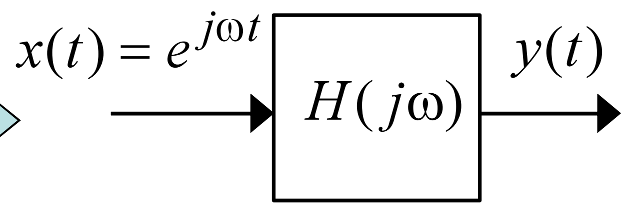
Review of Fourier Series

- Why Fourier series? Want to make analysis simple.
- Complex exponents are the eigenfunctions of LTI systems,

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} h(\tau) x(t-\tau)\, d\tau$$

$$x(t) = e^{j\omega t} \Rightarrow y(t) = H(j\omega) e^{j\omega t}$$



- x(t) is presented as linear combination of complex exponents (with different frequencies)
- LTI system response is the same linear combination of individual responses!

Periodic Signals

- Periodic signals are very important class of signals (widely used), where smallest T is a period,

$$x(t) = x(t+T), \text{ for all } t$$

- Examples: $\cos(\omega_0 t)$ & $e^{j\omega_0 t}$. Period $T = 2\pi / \omega_0$
- Introduce a set of harmonically-related complex exponents,

$$\phi_n(t) = e^{jn\omega_0 t} = e^{jn\frac{2\pi}{T}t}, \quad n = 0, \pm 1, \pm 2, \ldots$$

DC (n = 0)   1st harmonic (n = ±1)

- Construct a periodic signal,

$$x'(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t}$$

Fourier Series of Periodic Signal

- Can $x'(t)$ be made the same as $x(t)$?
- Yes, by adjusting c_n ,

$$c_n = \frac{1}{T}\int_T x(t) e^{-j2\pi\frac{n}{T}t} dt \quad \Rightarrow \quad x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t}, \omega_0 = \frac{2\pi}{T}$$

- $\{c_n\}$ – Fourier series coefficients (or spectral coefficients, or discrete spectrum of the signal)
- c_0 – DC component or average value of x(t),

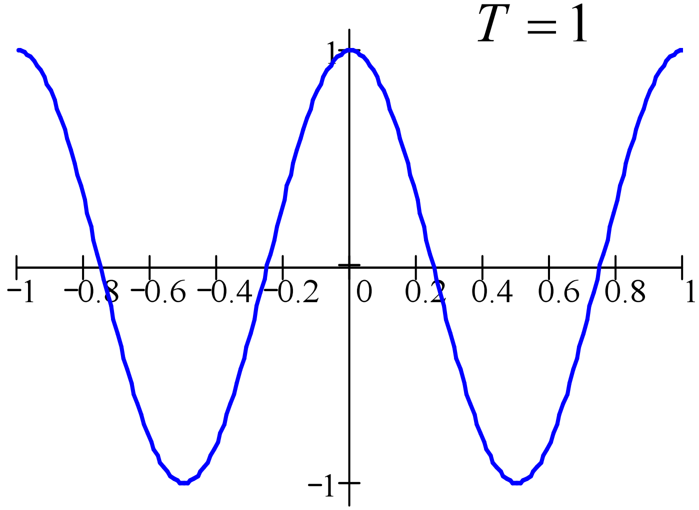
$$c_0 = \frac{1}{T}\int_T x(t) dt$$

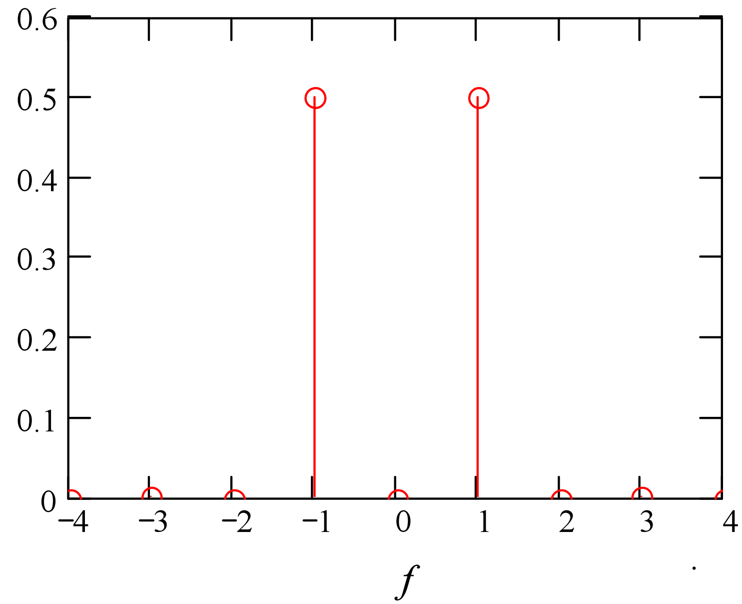
Example of Fourier Series

$$x(t) = \cos(\omega_0 t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

$$c_1 = \frac{1}{2}, c_{-1} = \frac{1}{2},$$
$$c_k = 0, k \neq \pm 1$$

$T = 1$



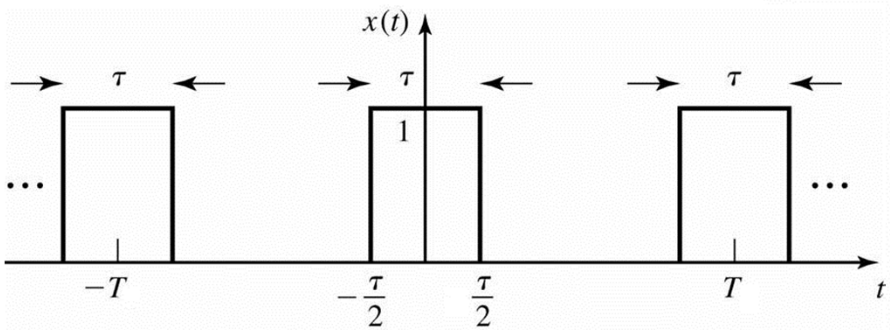


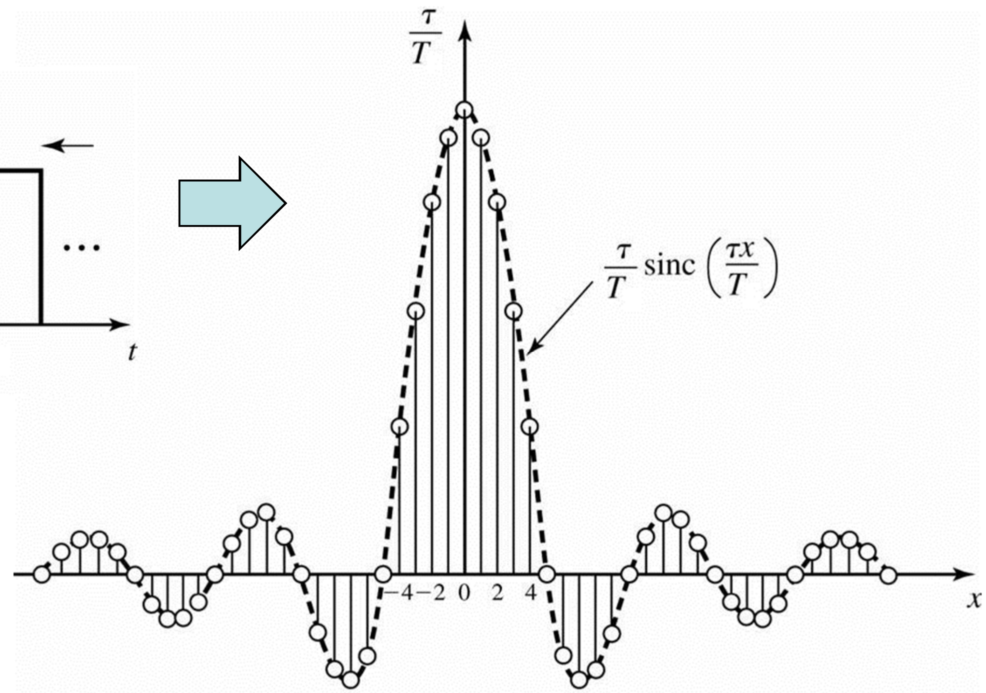
Example of Fourier Series

$$x(t) = \begin{cases} 1, |t| < \tau/2 \\ 0, \tau/2 < |t| < T/2 \end{cases}$$

$$c_n = \frac{\tau}{T}\,\mathrm{sinc}\left(\frac{n\tau}{T}\right),$$

$$\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

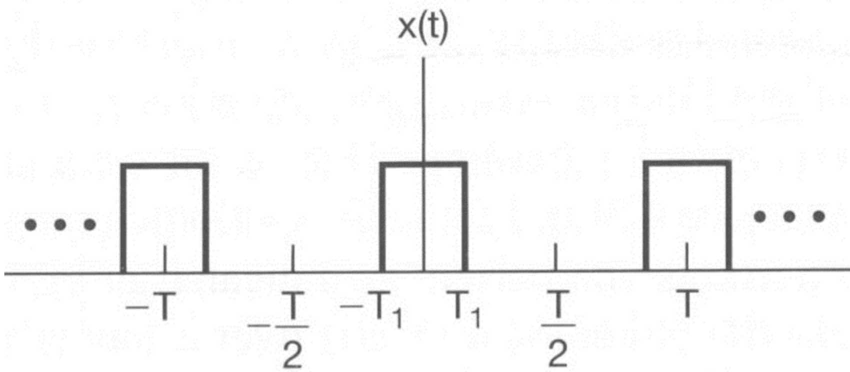




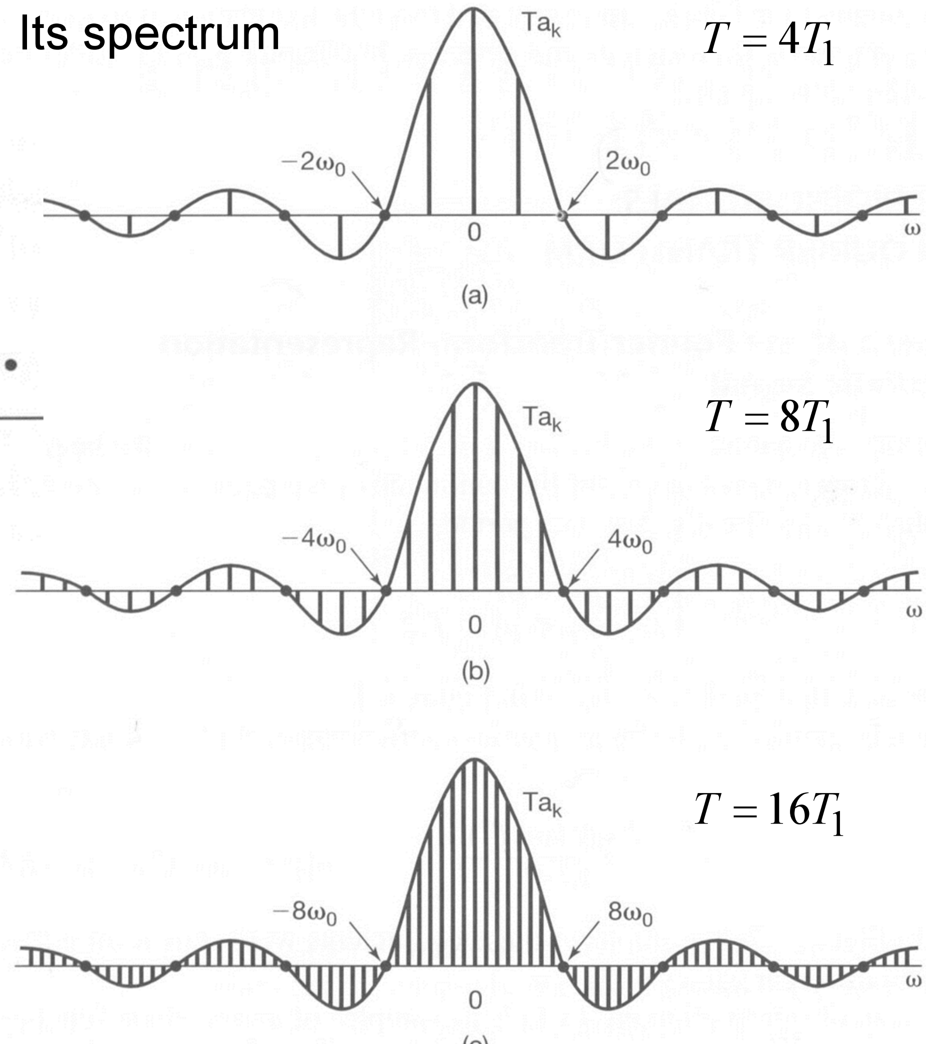
Q.: How does c_n scale with the pulse amplitude? Duration? Period?

Example of Fourier Series

Periodic signal



Its spectrum



$T = 4T_1$

$T = 8T_1$

$T = 16T_1$

A.V. Oppenheim, A.S. Willsky, Signals and Systems, 1997.

Q.: How does c_n scale with the pulse amplitude? Duration? Period?

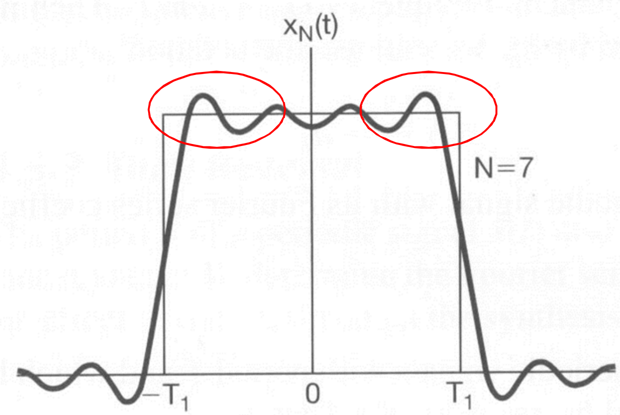
Convergence of Fourier Series

- Dirichlet conditions:
  - x(t) must be absolutely integrable (finite power)

$$\int_T |x(t)| dt < \infty$$

  - x(t) must be of bounded variation; that is the number of maxima and minima during a period is finite
  - In any finite interval of time, there are only a finite number of discontinuities, which are finite.
- Dirichlet conditions are only sufficient, but are not necessary.
- All physically-reasonable (practical) signals meet these conditions.

Gibbs Phenomenon



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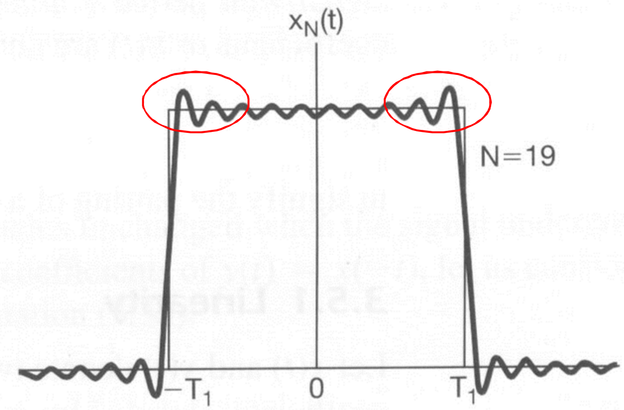

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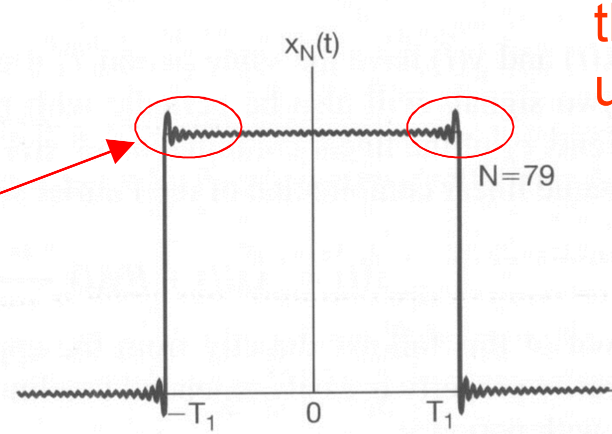
Q.: reproduce these graphs using a computer



(e)

increasing the number of terms does not decrease the ripple maximum!

Fourier Series of Real Signals

- For a real signal, $\mathrm{Im}\{x(t)\} = 0 \Rightarrow c_{-n} = c_n^*$
- Then obtain the trigonometric Fourier series,

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right], \quad \omega_0 = \frac{2\pi}{T}$$

$$a_n = 2\,\mathrm{Re}\{c_n\} = \frac{2}{T}\int_T x(t)\cos(n\omega_0 t)dt, \quad b_n = -2\,\mathrm{Im}\{c_n\} = \frac{2}{T}\int_T x(t)\sin(n\omega_0 t)dt$$

- Another form of it is

$$x(t) = x_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \varphi_n)$$

$$A_n = |c_n| = \sqrt{a_n^2 + b_n^2}, \quad \varphi_n = -\arg(c_n) = -\tan^{-1}(b_n / a_n)$$

Properties of Fourier Series

- Linearity: $$\mathrm{F}\left[\alpha x_1(t)+\beta x_2(t)\right]=\alpha\mathrm{F}\left[x_1(t)\right]+\beta\mathrm{F}\left[x_2(t)\right]$$
- Time shifting: $$x(t)\overset{F}{\longleftrightarrow}c_n \Leftrightarrow x(t-t_0)\overset{F}{\longleftrightarrow}e^{-jn\omega_0 t_0}c_n$$
- Time reversal: $$x(t)\overset{F}{\longleftrightarrow}c_n \Leftrightarrow x(-t)\overset{F}{\longleftrightarrow}c_{-n}$$
- Time scaling: $$x(\alpha t)=\sum_{n=-\infty}^{+\infty}c_n e^{jn(\alpha\omega_0)t}$$

Q.: prove these properties

Properties of Fourier Series

- Multiplication: $x(t)y(t) \overset{F}{\longleftrightarrow} \sum_{k=-\infty}^{\infty} c'_k c''_{n-k}$
- Convolution: $\int_T x(\tau)y(t-\tau)d\tau \overset{F}{\longleftrightarrow} Tc'_n c''_n$
- Differentiation: $\frac{dx(t)}{dt} \overset{F}{\longleftrightarrow} jn\omega_0 c_n$
- Integration: $\int_{-\infty}^{t} x(\tau)d\tau \overset{F}{\longleftrightarrow} \frac{c_n}{jn\omega_0}$, for $c_0 = 0$

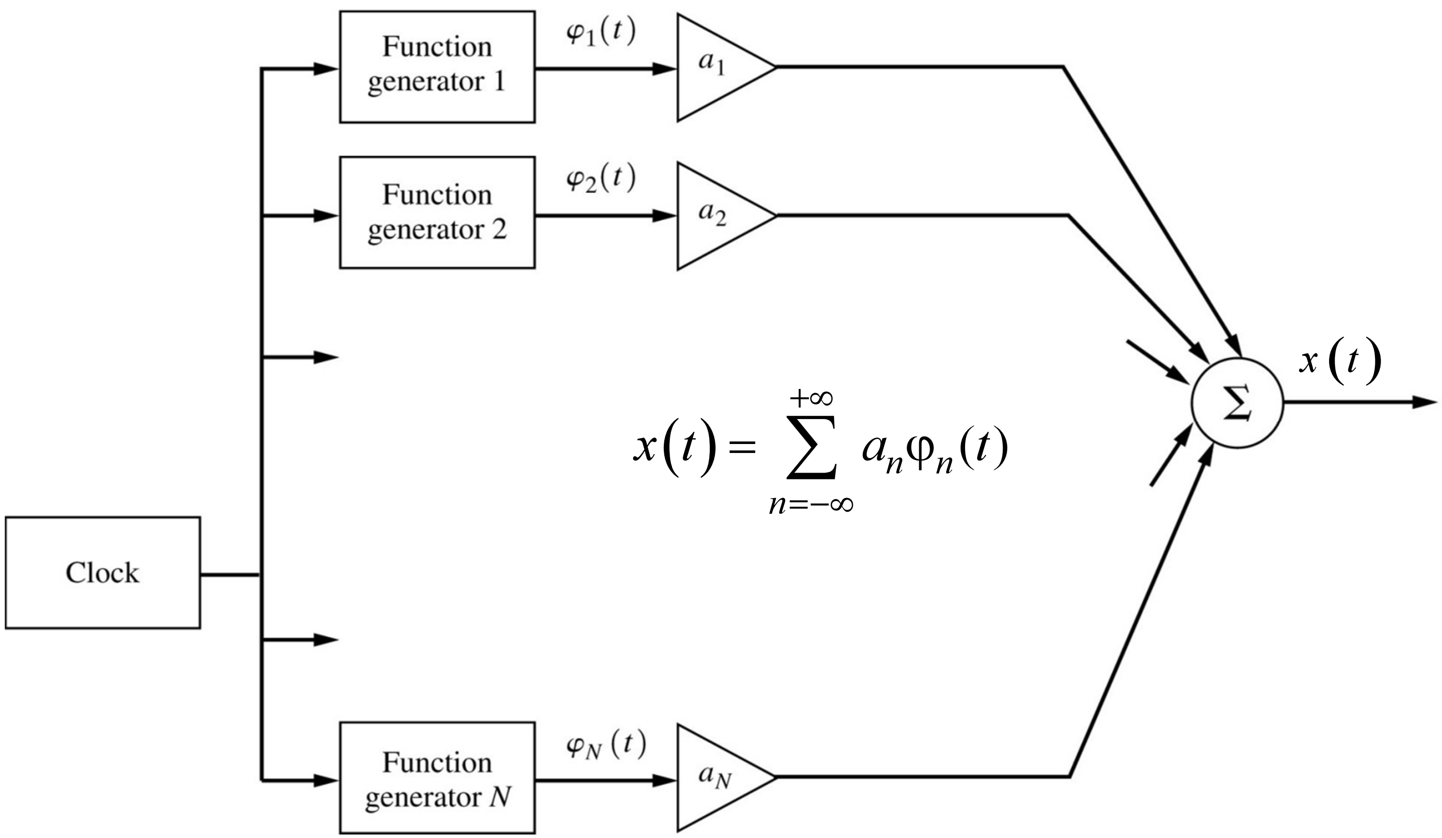
Q.: prove these properties

Properties of Fourier Series

- Real x(t): $$c_{-n} = c_n^*$$
- Real & even x(t): $$c_{-n} = c_n, \mathrm{Im}\{c_n\} = 0$$
- Real & odd x(t): $$c_{-n} = -c_n, \mathrm{Re}\{c_n\} = 0$$
- Parseval's Theorem: $$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |c_n|^2$$

Q.: prove these properties

Signal Synthesis via FS



$$x(t) = \sum_{n=-\infty}^{+\infty} a_n \varphi_n(t)$$

Couch, Digital and Analog Communication Systems, Seventh Edition.

Summary

- Review of Fourier series
- Periodic signals & complex exponents
- Series expansion of a periodic signal
- Trigonometric form of Fourier series
- Properties of Fourier series

- **Reading:** the Couch text, Sec. 2.1-2.5; Oppenheim & Willsky text, Sec. 3.0-3.5. Study carefully all the examples (including end-of-chapter study-aid examples), make sure you understand and can solve them with the book closed.
- Do some end-of-chapter problems. Students' solution manual provides solutions for many of them.