

CO 250 Assignment 1 – Spring 2012

Solutions

Problem 1: Linear Algebra Review

For each of the following statements, either prove that the statement is true, or provide a counterexample that shows that the statement is false.

- (a) For two 2×2 matrices A and B , $(AB)^T = A^T B^T$. (Note: A^T denotes the transpose of A).
- (b) For two vectors a and b , if $a \leq b$ and $b \leq a$ then $a = b$.
- (c) For three vectors a , b and c , if $a \leq b$ and $b \leq c$ then $a \leq c$.
- (d) Suppose A is an $m \times n$ matrix whose rank is less than m . Then there exists a nonzero vector x such that $Ax = \mathbf{0}$.
- (e) Let a and b be length n vectors. What is the dimension of the matrix ab^T ? What is its rank?
- (f) Suppose A is a 17×89 matrix. Then A can have 27 linearly independent columns.
- (g) Suppose A is a 3×4 matrix. Let A_1 be the 3×2 submatrix consisting of the first two columns of A , and A_2 the 3×2 submatrix consisting of the third and fourth columns of A . If x_1 and x_2 are vectors of length 2, and $x = (x_1^T, x_2^T)^T$, then $Ax = A_1x_1 + A_2x_2$.

Solution:

- (a) False. Take

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

Now simple matrix arithmetic gives

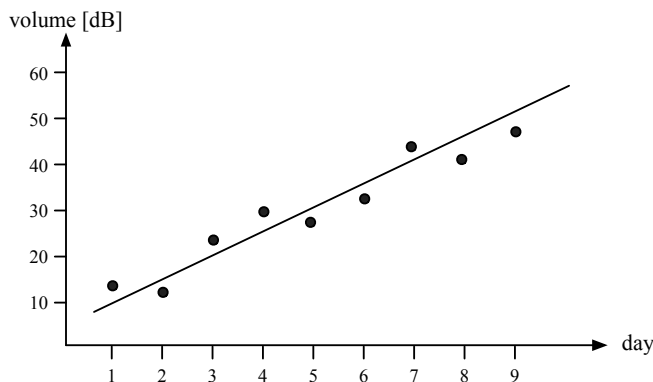
$$(AB)^T = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad A^T B^T = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}$$

- (b) True. For two vectors a and b , $a \leq b$ iff $a_i \leq b_i$ for all $1 \leq i \leq n$. So if $a \leq b$ and $b \leq a$, then for each i , $a_i \leq b_i$ and $b_i \leq a_i$, implying $a_i = b_i$. Thus $a = b$.
- (c) True. Again, for each i we have $a_i \leq b_i$ and $b_i \leq c_i$ implying that $a_i \leq c_i$. Thus $a \leq c$.
- (d) False. Consider a linearly independent set Y of vectors of length m , with $|Y| < m$. Setting A to be the matrix obtained by taking the elements of Y as columns, we have an $m \times n$ matrix where $n = |Y|$. Now $m > n$, n is the rank of A , and there is no non-zero vector x for which $Ax = \mathbf{0}$.
- (e) The matrix $M = ab^T$ is an $n \times n$ matrix with rank at most 1. If a and b are both different from $\mathbf{0}$, then M has rank 1 because every row of M is a scalar multiple of b (and every column is a scalar multiple of a). Otherwise M is the zero matrix, and has rank zero.
- (f) False. The size of a maximum set of linearly independent columns of A is equal to the size of a maximum set of linearly independent rows of A . Since A has 17 rows, a set of linearly independent columns of A has size at most 17.
- (g) True. The expression Ax is a linear combination of the columns of A with coefficients given by x . If the columns of A are a_1, \dots, a_4 and the entries of x are x^1, \dots, x^4 , then $Ax = a_1x^1 + \dots + a_4x^4 = A_1x_1 + A_2x_2$.

Problem 2: LP Models

For about two years now, residents of Winston have been plagued by an ominous noise at night. The source of the noise is unclear, but it appears to be coming from across the river that flows through town, from the city of Steeltown. What is worse, the noise seems to be getting louder every night! In order to prove this fact, the residents of Winston are taking noise measurements on a nightly basis. The following table and graph show the data for 9 consecutive measurements.

day i	1	2	3	4	5	6	7	8	9
noise n_i [dB]	13	11	21	29	27	30	41	37	42



The picture seems to show that the increase in noise level is linear; i.e., that there is a line (as depicted in the figure) that explains the noise developments in Winston; i.e., there are $a, b \in \mathbb{R}$ such that the noise on day i is roughly $ai + b$. But how do we choose a and b ? A savvy resident writes down the following mathematical program for this task:

$$\min \sum_{i=1}^9 |(ai + b) - n_i|, \quad (1)$$

where n_i is the measured noise level on day i , $|\cdot|$ denotes the absolute value function. Mathematical program (1) has two variables, a and b , and attempts to choose these such that the total absolute error for the observed data with respect to the linear function $ax + b$ is minimized.

Note that (1) is not a linear program: the absolute value function is not linear! But our savvy friend from Winston claims that one can use linear programming to formulate this problem. Can you help?

Solution: First note that we can replace $|(ai + b) - n_i|$ in the objective function with a new variable y_i , and the constraints $y_i \leq |(ai + b) - n_i|$ and $y_i \geq |(ai + b) - n_i|$. Moreover, since we are trying to minimize the value of the objective function, we only need the constraints $y_i \geq |(ai + b) - n_i|$. This gives us the more familiar looking mathematical program:

$$\begin{aligned} \min \quad & \sum_{i=1}^9 y_i \\ \text{s.t.} \quad & y_i \geq |(ai + b) - n_i| \quad 1 \leq i \leq n. \end{aligned}$$

It remains to remove the absolute value expressions from the constraint set. Note that if $x \geq |\alpha|$, then $x \geq \alpha$ and $x \geq -\alpha$. So we can replace $y_i \geq |(ai + b) - n_i|$ with the constraints $y_i \geq (ai + b) - n_i$ and $y_i \geq -(ai + b) + n_i$. So our linear program is:

$$\begin{aligned} \min \quad & \sum_{i=1}^9 y_i \\ \text{s.t.} \quad & y_i \geq (ai + b) - n_i \quad 1 \leq i \leq n \\ & y_i \geq -(ai + b) + n_i \quad 1 \leq i \leq n. \end{aligned}$$

Problem 3: LP Models

A company must meet (on time) the following demands:

Quarter	1	2	3	4
Demand (in units)	30	20	40	30

Each quarter, up to 27 units can be produced with regular-time labour, at a cost of \$40 per unit. During each quarter, an unlimited number of units can be produced with overtime labour, at a cost of \$60 per unit. Of all units produced, 20% are unsuitable and cannot be used to meet demands. The company is able to store excess units for future use at a cost of \$ 15 per unit per quarter.

Formulate an LP that can be used to minimize the total cost of meeting the next four quarters' demands. Assume that 20 usable units are available at the beginning of quarter 1.

Solution: We define the following decision variables:

$$\begin{aligned}x_i &= \# \text{ of units produced with regular labour during quarter } i, i = 1, 2, 3, 4. \\y_i &= \# \text{ of units produced with overtime labour during quarter } i, i = 1, 2, 3, 4. \\z_i &= \# \text{ of units held from quarter } i \text{ to the next quarter } i = 1, 2, 3.\end{aligned}$$

First, the total cost is

$$40 \left(\sum_{i=1}^4 x_i \right) + 60 \left(\sum_{i=1}^4 y_i \right) + 15 \left(\sum_{i=1}^3 z_i \right) \quad (2)$$

Each quarter, at most 27 units can be produced using regular-time labour.

$$x_i \leq 27 \quad i = 1, 2, 3, 4. \quad (3)$$

Next, we compute the number of usable units left over by the end of quarter 1. We have 20 usable units to start with. $(x_1 + y_1)$ units are produced during the quarter, but only 80% of them are usable. There is a demand of 30. Thus,

$$20 + 0.8(x_1 + y_1) - 30 = z_1. \quad (4)$$

Similar, for quarter 2, we start with z_1 usable units, produce $0.8(x_2 + y_2)$ usable ones, while the demand is 20 units. Hence,

$$z_1 + 0.8(x_2 + y_2) - 20 = z_2. \quad (5)$$

By the same rationale, the constraint for quarter 3 is

$$z_2 + 0.8(x_3 + y_3) - 40 = z_3. \quad (6)$$

Finally, for quarter 4, we just need to make sure that the demand is met, and need not worry about left over units.

$$z_3 + 0.8(x_4 + y_4) \geq 30. \quad (7)$$

We also need to make sure that all our variables are nonnegative.

$$x_i, y_i \geq 0 \quad i = 1, 2, 3, 4; \quad z_i \geq 0 \quad i = 1, 2, 3. \quad (8)$$

Finally, our LP formulation is

$$\begin{aligned}\min & \quad (2) \\s.t. & \quad (3), (4), (5), (6), (7), (8).\end{aligned}$$

Remark: Some may wonder why we don't need a constraint to make sure the demand is met for quarters 1-3, like we did for quarter 4. This is because the variables z_i 's are enforced to be nonnegative, which implies that at the end of each quarter, after the demand is met, we still have a nonnegative number of usable units left.

Problem 4: IP Models

At the local coffee shop, which is open weekdays only, workers are scheduled for four weekdays on followed by one day off and this schedule is repeated every five weekdays. Thus, a worker has the same day off each week. The demand for workers is given in the table below. This demand must be met or exceeded on each day.

Day	Mon	Tues	Wed	Thurs	Fri
Demand	3	5	9	2	7

Formulate an integer program that will minimize the total number of workers needed to meet the daily demands.

Solution: Define decision variables x_i for $1 \leq i \leq 5$, where x_i denotes the number of workers scheduled with day i off (Monday is 1, Tuesday is 2, etc.).

On day i , all of the workers are present, except those with day i off. So the total number of workers working is $\sum_{j \neq i} x_j$. For each i we must ensure that this sum meets the corresponding demand d_i . So for each i we need the constraint

$$\sum_{j \neq i} x_j \geq d_i.$$

Moreover, we cannot hire a negative number of workers, nor a non-integral number of workers for any shift.

Subject to these constraints we want to minimize the total number of workers employed, $\sum_{i=1}^5 x_i$. Thus we can model this problem with the following integer program:

$$\begin{aligned} \min \quad & \sum_{i=1}^5 x_i \\ \text{s.t.} \quad & \sum_{j \neq i} x_j \geq d_i \quad 1 \leq i \leq 5 \\ & x_i \geq 0 \\ & x_i \in \mathbb{Z}. \end{aligned}$$

Problem 5: IP Models

A paper mill produces rolls of paper of a standard width of three meters (=300cm). Customers want to purchase paper rolls of smaller widths, and the mill has to cut such rolls from the 3 meter rolls. For example, a 3m roll can be cut into 3 rolls of 57cm, and one roll of 125cm; the remaining 4cm go to waste. Let us consider an order of

- (1) 97 rolls of width 135cm,
- (2) 610 rolls of width 108cm,
- (3) 395 rolls of width 93cm, and
- (4) 211 rolls of width 42cm.

Formulate an integer program that determines the smallest number of 3m rolls needed to satisfy this order, and the way in which they need to be cut. **Hint:** A *cutting pattern* is of the form (a_1, a_2, a_3, a_4) where a_i is a non-negative integer such that

$$a_1 135 + a_2 108 + a_3 93 + a_4 42 \leq 300.$$

One way to formulate your problem is to have one variable for each pattern.

Solution: Following the hint, we define a *cutting pattern* to be a 4-tuple $a = (a_1, a_2, a_3, a_4)$ so that each a_i is a non-negative integer, and

$$135a_1 + 108a_2 + 93a_3 + 42a_4 \leq 300.$$

Let \mathcal{A} denote the set of all cutting patterns. For each $a \in \mathcal{A}$, let x_a be the decision variable denoting the number of three meter rolls to cut with cutting pattern a .

For each of the specified widths, we need to ensure that enough rolls are cut to meet the demand. For instance, we need 97 rolls of width 135cm. So we must satisfy the constraint:

$$\sum_{a \in \mathcal{A}} a_1 x_a \geq 97.$$

This is a linear constraint as the value a_1 is constant.

We want to minimize the total number of three meter rolls used. So we want to minimize the total number of cuts made, $\sum_{a \in \mathcal{A}} x_a$.

Adding the non-negativity and integrality constraints on the x_a , we obtain the final integer program:

$$\begin{array}{ll} \min & \sum_{a \in \mathcal{A}} x_a \\ \text{s.t.} & \sum_{a \in \mathcal{A}} a_1 x_a \geq 97 \\ & \sum_{a \in \mathcal{A}} a_2 x_a \geq 610 \\ & \sum_{a \in \mathcal{A}} a_3 x_a \geq 395 \\ & \sum_{a \in \mathcal{A}} a_4 x_a \geq 211 \\ & x_a \geq 0 \\ & x_a \in \mathbb{Z}. \end{array}$$