Lecture 1

Algebra Review. In this course, we expect you to be comfortable with the algebra prerequisites given in the course description. I'll give a review this class.

Sets. We let $\mathbb{R}$ be the real numbers and let $\mathbb{Z}$ be the integers. We are often interested in intervals:

- $[100, 300]$ means all the real numbers that are between 100 and 300 inclusively.
- $[100, 300)$ means all the real numbers that are between 100 and 300 including 100 but not including 300.
- $(100, 300)$ means all the real numbers that are strictly between 100 and 300.
- $[100, \infty)$ means all the real numbers that are greater than or equal to 100.

Note: ∞ is not a number and so is *never* included in an interval.
Pictorially:

Example 1. *Draw a diagram of the intervals*

(1) $(-\infty, 12]$

(2) $[-7, -3)$

We have more general notation for *sets*. For finite sets, we just enumerate items:

$$S = \{1, 3, 4, 12\}.$$

We sometimes write infinite sets this way if the pattern is obvious:

$$S = \{2, 4, 6, 8, \ldots\},$$

but this can be misleading. For this reason, we have some more general notation that allows us to avoid these confusing situations. This more general notation looks a lot like:

$$S = \{x \in A : \text{ some condition on } x\}.$$

For example, we might write

$$S_1 = \{x \in \mathbb{R} : x > 5\} = (5, \infty)$$
$$S_2 = \{x \in \mathbb{R} : x^2 < 4\} = (-2, 2)$$

$$S_3 = \{x \in \mathbb{N} \,:\, x \text{ is prime.}\} = \{2, 3, 5, 7, 11, \ldots\}.$$

Using this sort of notation, we might write $\{2, 4, 6, 8, \ldots\}$ as

$$S = \{2x \,:\, x \in \mathbb{N}\}.$$

There are some very common ways to combine sets that get special notation:

$$S \cap U = \{x \,:\, x \in S \text{ and } x \in U\}$$
$$S \cup U = \{x \,:\, x \in S \text{ or } x \in U\}$$

For example,

Example 2. *Let $S = \{1, 2, 3, 5\}$ and $U = \{1, 2, 12\}$. Calculate $S \cup U$ and $S \cap U$:*

$$S \cup U = \{1, 2, 3, 5, 12\}$$
$$S \cap U = \{1, 2\}.$$

Polynomials. Tools:

Quadratic Formula for solving $ax^2 + bx + c = 0$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Difference of Squares for factoring $x^2 - a^2$:

$$x^2 - a^2 = (x - a)(x + a)$$

Example 3. *Solve for x.*

$$-x^2 + 15x - 20 = 3x - 44$$

Solving Inequalities.

Example 4. *The price per unit, p, of a product is related to the demand, x, for the product by*

$$p = 120 - 0.03x.$$

Given that price and demand are both nonnegative, what values of p and x are possible?

We would like to find all pairs (p, x) that satisfy both:

$$p = 120 - 0.03x \geq 0$$
$$x \geq 0.$$

To solve this, we notice that $p = p(x)$ is a decreasing function of x. Thus, the allowed values of x form a single interval, so it is enough to solve the equation $p = 0$. That equation is simple:

$$\{0 = 120 - 0.03x\} \Longrightarrow \{x = 4000\}.$$

Thus, the solutions are

$$S = \{(120 - 0.03x, x) \,:\, x \in [0, 4000]\}.$$

Example 5. *The profit, P, of producing a product is related to the price per unit, u, by*

$$P = -15u^2 + 105u - 150.$$

For what values of u is the profit positive?

Notice that, unlike the previous example, P is not a decreasing or increasing function of u! Nonetheless, we start by solving $P(u) = 0$:

$$\begin{aligned} u &= \frac{-105 \pm \sqrt{(105)^2 - 4(-15)(-150)}}{30} \\ &= \frac{-105 \pm 45}{-30}. \end{aligned}$$

Thus, the solutions are $\{2, 5\}$. However, we aren't don't yet - how do these solutions relate to our inequality?

If we draw $P(u)$, we notice that $P(u) < 0$ for u very large or very small. Thus, we have

$$\{u \,:\, P(u) \geq 0\} = [2, 5].$$

Absolute Value. If a is a real number, then the **absolute value** of a is defined as

$$|a| = \begin{cases} a \text{ if } a \geq 0 \\ -a \text{ if } a \leq 0 \end{cases}$$

The absolute value of a number is like the "positive part" or "magnitude" of the number. If $c \geq 0$ is a positive constant, then

$$\begin{aligned} |x - a| \leq c \quad &\Leftrightarrow \quad -c \leq x - a \leq c \\ &\Leftrightarrow \quad a - c \leq x \leq a + c \end{aligned}$$

The other inequality $|x - a| \geq c$ has two solutions:

$$x - a \geq c \quad \text{or} \quad x - a \leq -c$$

Examples

(1) Solve $|25 - x| \geq 20$.

<u>*solution:*</u> By the second remark above, we have that either

$$\begin{array}{rcl} 25 - x \geq 20 & \text{or} & 25 - x \leq -20 \\ -x \geq -5 & & -x \leq -45 \\ x \leq 5 & & x \geq 45 \end{array}$$

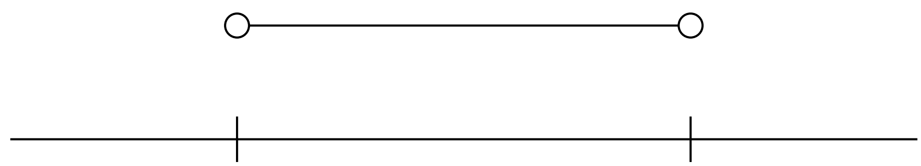
The set of solutions is represented below:

(2) Solve $\left|1 - \frac{2x}{3}\right| < 1$.

<u>*solution:*</u>

$$\left|1 - \frac{2x}{3}\right| < 1$$
$$-1 < 1 - \frac{2}{3}x < 1$$
$$-2 < -\frac{2}{3}x < 0$$
$$3 > x > 0$$

Thus the set of solutions is the interval $(0, 3)$ and is represented as below:



Exponents. Rules and Notation:

(1) $x^n = x \times x \times \ldots \times x$
(2) $x^0 = 1$
(3) $x^{-n} = \frac{1}{x^n}$
(4) $x^{1/n} = \sqrt[n]{x}$
(5) $x^n x^m = x^{n+m}$
(6) $\frac{x^n}{x^m} = x^{n-m}$
(7) $(xy)^n = x^n y^n$
(8) $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$
(9) $(x^n)^m = x^{nm}$

Example: Simplify

$$\left(\frac{x^5\sqrt{x}}{\sqrt[3]{x}^7}\right)^{-2}$$

Fractions and Rationalization. To rationalize an expression,

(1) If $\sqrt{a}$ is in the denominator, multiply by $\frac{\sqrt{a}}{\sqrt{a}}$.
(2) If $\sqrt{a} - \sqrt{b}$ is in the denominator, multiply by $\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}}$.
(3) If $\sqrt{a} + \sqrt{b}$ is in the denominator, multiply by $\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}}$.

Note that in each case the term that we are multiplying by is equal to 1, therefore we are not changing the expression, just altering its appearance.

<u>Examples:</u>

(1) Simplify $\frac{13}{6+\sqrt{10}}$.

<u>*solution:*</u>

$$\frac{13}{6+\sqrt{10}} = \frac{13}{6+\sqrt{10}} \cdot \left(\frac{6-\sqrt{10}}{6-\sqrt{10}}\right) = \frac{13(6-\sqrt{10})}{6^2 - \sqrt{10}^2}$$

$$= \frac{13(6-\sqrt{10})}{26} = \frac{(6-\sqrt{10})}{2}.$$

(2) Simplify $\frac{(x-1)}{\sqrt{x}+x}$.

<u>*solution:*</u>

$$\frac{(x-1)}{\sqrt{x}+x} = \frac{(x-1)}{\sqrt{x}+x} \cdot \left(\frac{\sqrt{x}-x}{\sqrt{x}-x}\right)$$
$$= \frac{(x-1)(\sqrt{x}-x)}{\sqrt{x}^2 - x^2} = \frac{(x-1)(\sqrt{x}-x)}{x-x^2} = \cdots$$
$$\cdots = \frac{(x-1)(\sqrt{x}-x)}{x(x-1)} = \frac{\sqrt{x}-x}{-x}$$
$$= \frac{x-\sqrt{x}}{x} = 1 - \frac{\sqrt{x}}{x} = 1 - x^{-1/2}.$$

Rational Functions. If $P(x), Q(x)$ are polynomials, then $f(x) = \frac{P(x)}{Q(x)}$ is called a *rational function.* In this class, you will need to be able to simplify rational functions - that is, remove common factors from $P(x)$ and $Q(x)$:

Example 6. *Simplify the following functions:*

$$f(x) = \frac{(x-1)(x-2)}{(x-1)(x-3)}$$
$$g(x) = \frac{x^2-2x-8}{x^2-16}.$$

The first is easy:

$$f(x) = \frac{x-2}{x-3}.$$

To do the second, we must factor the numerator and denominator. By the quadratic formula, the solution to $x^2-2x-8=0$ are

$$x = \frac{2 \pm \sqrt{4+32}}{2}$$
$$= 1 \pm 3.$$

Thus,

$$x^2 - 2x - 8 = (x-4)(x+2).$$

Similarly,

$$x^2 - 16 = (x-4)(x+4).$$

Thus, we can write:

$$g(x) = \frac{(x-4)(x+2)}{(x-4)(x+4)}$$
$$= \frac{x+2}{x+4}.$$

Example 7 (Short Recap Quiz)**.** *(1) Solve $x^2 - 2x \geq 1$.*

(2) Solve $|x-5| \geq |x| + 1$.

(3) Simplify $\frac{x^2-6x+9}{x^2-9}$.

Functions. A *function* is a mathematical relationship between two variables (called the *independent variable* and the *dependent variable*) such that each value of the independent variable corresponds to a unique value of the dependent variable.

The *domain* of a function is the set of all the possible allowed values of the independent variable.

The *range* of a function is the set of all the values of the dependent variable which correspond to some value of the independent variable in the domain.

Often mathematicians will use x to denote the independent variable, y to denote the dependent variable, and f to denote the function so $y = f(x)$.

Question: Where do functions come from?

They come from our analysis of the perceived relationship between two variables (such as profit and number of units sold). When a certain relationship is simple and well-studied, we may be able to find a function that exactly describes the relationship between the two variables. If the relationship is complex, then statisticians can collect data on the relationship and create a function that seems to be an acceptable fit to the data. There is often a tradeoff between the simplicity and the accuracy of the function.

Example 8. *The profit, P, of producing x units of a product is given by*

$$P(x) = 400x - \sqrt{40x - 1000} - 60000.$$

(1) What is the domain of this function?
(2) What is the profit when 275 units are produced?

To find the domain, we note:

- *We can only produce a nonnegative number of units. Thus, $x \geq 0$.*
- *The term $\sqrt{40x - 1000}$ only makes sense if $x \geq 25$.*
- *The profit is negative for some values of x - but that is fine!*

Thus, the best domain is $[25, \infty)$.

When 100 units are produced,

$$\begin{aligned} P(100) &= (400)(275) - \sqrt{11000 - 1000} - 60000 \\ &= 49900. \end{aligned}$$

Example 9. *Find the domain of the function*

$$f(x) = \frac{x^2}{x-1}$$

This function makes sense when $x \neq 1$.

Composite Functions. Given two functions f and g, the *composite*, $f \circ g$, is a new function whose values are $f(g(x))$.

Example 10. *Let $f(x) = 1 + x^2$ and $g(x) = 2x - 1$. Find $f \circ g$ and $g \circ f$.*

We have

$$(f \circ g)(x) = f(g(x)) = (1 + (2x - 1)^2)$$

and

$$(g \circ f)(x) = g(f(x)) = 2(1 + x^2) - 1.$$

Example 11. *The profit, $P(x)$, of selling x units of a product is given by*

$$P(x) = 5x - \sqrt{4x - 100} - 6000.$$

If the number of units sold, x, depends on the price per unit, q as

$$x(q) = \frac{50000}{q},$$

find P as a function of q.

We have

$$P(x(q)) = 5\frac{50000}{q} - \sqrt{4\frac{50000}{q} - 100} - 6000.$$

Inverses. Two functions f and g are *inverses* if $f(g(x)) = x$ and $g(f(x)) = x$ for all x in the domain of g and f respectively. Often the inverse of a function f is denoted f^{-1}.

(Roughly speaking, f^{-1} undoes what f did to x.)

Ex: $f(x) = x^3$ and $g(x) = x^{1/3}$ are inverses.

Ex: $f(x) = x^2$ has no inverse but $f(x) = x^2, x \geq 0$ has inverse $f^{-1}(x) = \sqrt{x}$.

We can find the inverse by solving for the independent variable and then appropriately renaming the variables.

Example 12. ***Question:*** *Find the inverse of $f(x) = \frac{2x-1}{x+7}$.*

Answer: *We solve $y = \frac{2x-1}{x+7}$ for x:*

$$y = \frac{2x - 1}{x + 7},$$

so

$$2x - 1 = xy + 7y,$$

so

$$x(2 - y) = 7y + 1.$$

We conclude

$$x = \frac{7y + 1}{2 - y},$$

so

$$f^{-1}(x) = \frac{7x + 1}{2 - x}.$$

Graphs. Graphs give a pictorial representation of the relationship described by a function. You should be familiar with the graphs of

1) Linear Functions: $y = mx$ (lines)
2) Quadratic Functions: $y = x^2$ (parabolas)
3) Cubic Functions: $y = x^3$
4) Square Root Functions: $y = \sqrt{x}$
5) Absolute Value Functions: $y = |x|$
6) Hyperbolic Functions: $y = 1/x$

- Replacing x with $x - a$ moves the graph a units to the right.
- Replacing y with $y - b$ moves the graph b units up.

Example 13. ***Question:*** *Graph $y = |x - 3| - 4$.*

Answer: *Rewriting as $y + 4 = |x - 3|$ we see that this is like $y = |x|$ but shifted 3 units to the right and 4 units down.* ***In class****, we draw this on the board.*

Points of Intersection. **x-intercepts:** points at which the graph crosses the x-axis.
→ We find them by solving $f(x) = 0$.
y-intercepts: points at which the graph crosses the y-axis.
→ We find them by substituting $x = 0$.

Example 14. ***Question:*** *Find the x- and y-intercepts of $y = |x - 3| - 4$.*

Answer: *When $y = 0$, we have $|x - 3| = 4$, so $x \in \{-1, 7\}$. Thus, the y-intercepts are $\{-1, 7\}$.*

When $x = 0$, we have $y = |-3| - 4 = -1$, so the x-intercept is -1.

The *break-even point* is a number of units that must be sold for the total costs to be equal to the total revenue (i.e. the profit is 0).

Example 15. *Producing CDs requires an initial investment of \$1980 for studio time and then \$1.80 for each CD made. We sell the CDs for \$9 each. Sketch the graphs of the cost and revenue functions on the same axes. What is the break-even point?*

Lines. A *linear equation* is an equation that can be written in the form $y = mx + b$ (slope-y-intercept form) where m is the slope of the line and b is the y-intercept.

Given two points (x_1, y_1) and (x_2, y_2) on the line we can calculate the slope.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

The slope is how much y changes for each 1 unit increase in x.

Given a point (x_1, y_1) on the line and the slope m we can find the linear equation using

$$y - y_1 = m(x - x_1).$$

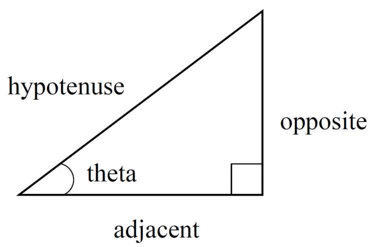
Example 16. *Suppose that the relationship between the demand, x, for a product and the price per unit, p is linear. If we can sell 1000 units at \$20 each and 3000 units at \$15 each, find the demand function.*

Using the point-intercept form, we have

$$(y - 20) = \frac{15 - 20}{3000 - 1000}(x - 1000).$$

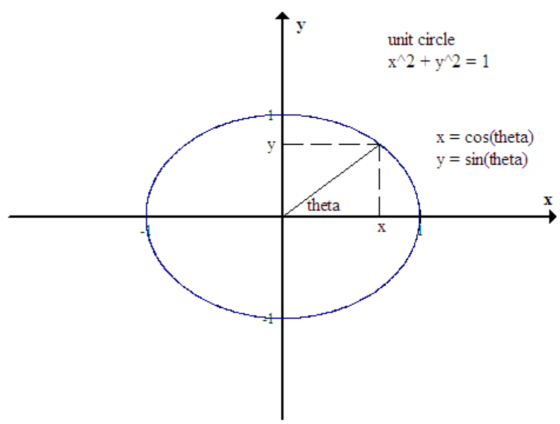
Trigonometric Functions. Recall the definitions of the trigonometric functions from a right-angled triangle.

$$\cos\theta = \frac{adjacent}{hypotenuse}$$
$$\sin\theta = \frac{opposite}{hypotenuse}$$
$$\tan\theta = \frac{opposite}{adjacent}$$



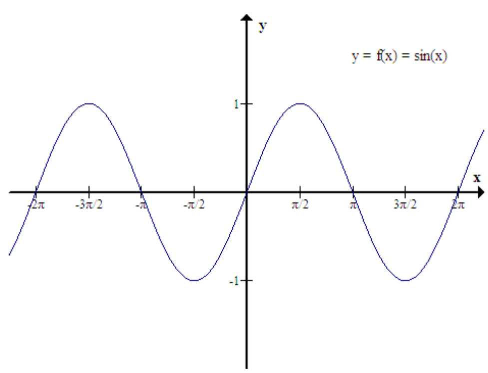
$$\sec\theta = \frac{1}{\cos\theta} = \frac{hypotenuse}{adjacent}$$
$$\csc\theta = \frac{1}{\sin\theta} = \frac{hypotenuse}{opposite}$$
$$\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta} = \frac{adjacent}{opposite}$$

Or, from the unit circle, $x^2 + y^2 = 1$.



$x = \cos\theta$ and $y = \sin\theta$.

The graph of $\sin(x)$ looks like:



You may recall from highschool that there are very many trig formulas. We don't focus on them in this course, but you should be familiar with them. The most important for this course are:

$$\begin{aligned}
\cos(x) &= \sin(x + \frac{\pi}{2}) \\
\sin^2(x) + \cos^2(x) &= 1 \\
\sin(x+y) &= \sin(x)\cos(y) + \cos(x)\sin(y) \\
\cos(x+y) &= \sin(x)\cos(y) - \cos(x)\sin(y) \\
\sin^2(x) &= \frac{1}{2}(1 - \cos(2x)) \\
\cos^2(x) &= \frac{1}{2}(1 + \cos(2x)) \\
\sin(x) + \sin(y) &= 2\sin(\frac{x+y}{2})\cos(\frac{x-y}{2}).
\end{aligned}$$

Calculus Preview. We'll talk about the *rate of change*, which motivated the development of calculus. The point of studying differential calculus is to study the relationship between the *value* of a function and *how quickly that value is changing.* I think this is normally easiest to understand by looking at the relationship between the *position* of an object and its *speed.*

Definition 0.1 (Average Speed)**.** *Let $f(x)$ be some function. Its average speed over the interval $[a, b]$ is given by*

$$v = \frac{f(b) - f(a)}{b - a}.$$

Example 17 (Car Travel)**.** ***Question:*** *I travel from Ottawa to Toronto, a distance of 450 kilometers, in 5 hours. What was my average speed?*

Answer: *Let $d(t)$ be the distance traveled after t hours. We calculate*

$$v = \frac{d(5) - d(0)}{5 - 0} = \frac{450 - 0}{5 - 0} = 90.$$

This seems like a very sensible way to measure speed for this example.

We give another example, illustrating the fact that an object's *average* speed may not always accurately reflect how quickly it is moving at any given moment.

Example 18 (Height of a Spring)**.** *We have a weight bobbing on the end of a spring. The height of that spring is given by the formula*

$$h(t) = 4 + \cos(\pi t),$$

where h is measured in feet and t is measured in seconds.[1]

How quickly is the spring moving? For our current definition, we need to say both the starting time and the end time. For starting $t = 0$, we get the following speeds:

We can see that the answer depends a huge amount on the time interval that we look at! However, it seems that small time intervals agree with our intuition pretty well: the weight "isn't moving" when we let go of it at time $t = 0$, and the "average speed" gets close to 0 as the interval becomes small.

[1]Note to physicists - I'm mostly choosing constants to make the equations easy to write down. They are normally not realistic.

Table 1. Speed

Interval	*Speed*
$(0, 200)$	*0*
$(0, 2)$	*0*
$(0, 1)$	*-2*
$(0, 0.75)$	*-2.28*
$(0, 0.5)$	*-2*
$(0, 0.25)$	*-1.17*
$(0, 0.1)$	*-0.49*
$(0, 0.01)$	*-0.049*
$(0, 0.001)$	*-0.0049*

We can try to look at the rate of change using equations:

Example 19. ***Question:*** *A ball falls from a window that is 50 meters up from the ground. For the first few seconds of its fall, its height in meters is given by the formula* $H(t) = -9.8t^2 + 58.8t + 50$. *Give a formula for the average speed of the ball over the time interval* $[1.5, 1.5+h]$.

Answer: *We plug into our earlier formula:*

$$v = \frac{H(1.5+h) - H(1.5)}{1.5+h-1.5}$$

$$v = \frac{9.8\left(-(1.5+h)^2 + (1.5)^2\right) + 58.8h}{h}$$

$$v = \frac{9.8\left(-3h - h^2\right) + 58.8h}{h}$$

$$= 29.4 - 9.8\,h.$$

So, that was an easy algebra exercise. However, we'll take a second to notice some things about this formula that seem important:

- *Our ball is accelerating, so the longer the interval* h*, the higher the average speed.*
- *For* h *very small, the speed is basically 29.4. This is suggestive that maybe the 'instantaneous' speed should be 29.4, even though we haven't really defined 'instantaneous' speed yet.*
- ***In class: We draw*** h***, then draw the line with slope 29.4 passing through the point*** $(1.5, h(1.5))$***. We point out that this line seems to 'kiss'*** h***. Such a line is called a tangent, and will be very important.***

None of these things are coincidences, and we'll get back to this soon.

Making this careful needs some more definitions. Unfortunately, the very first definition, the *limit*, is also the most difficult definition in the entire course. We'll get to this next class.

Additional Worked Problems

Recap of Techniques and Problems. In our first class, we saw:

(1) a review of prerequisite subjects, and
(2) an introduction to rate-of-change problems.

Short Quiz. Let's try a few questions:

- A car's position at time s is given by $x(s) = 2s + 3$. Calculate the average rate of change over the interval $(1, 2)$.
- Let $f(x) = x^2 - 4x + 8$. Find $\{x : f(x) \geq 2\}$.
- Find $\{x : |x - 1| \geq |x + 1|\}$.
- Simplify $\frac{x^2-2x+1}{x^2-1}$.

Further Examples.

Example 20 (Rate of Change Problem)**.** *The position of a weight on a spring is given by $x(s) = 2 + \cos(s)e^{-s}$. Calculate the average rate of change over the intervals $(0, \frac{\pi}{2})$ and $(0, 0.1)$.*

Plugging into a calculator,

$$v_1 = \frac{x(\frac{\pi}{2}) - x(0)}{\frac{\pi}{2}} = -\frac{2}{\pi}$$

and

$$v_1 = \frac{x(0.1) - x(0)}{0.1 - 0} \approx -6.34.$$

Example 21 (Inequalities 1)**.** *Find all values of $x \geq 1$ for which $x^2 - 30x + 125 \geq 0$.*

By the quadratic formula,

$$x = \frac{30 \pm \sqrt{900 - 500}}{2} = 15 \pm 10.$$

Thus, the solutions to $x^2 - 30x + 125 = 0$ are $x \in \{5, 25\}$. Drawing the picture, we have

$$\{x : x^2 - 30x + 125 \geq 0\} = (-\infty, 5] \cap [25, \infty).$$

Thus, the solution is $[1, 5] \cap [25, \infty)$.

Example 22 (Inequalities 1)**.** *Find all values of $x \leq 10$ for which $|x - 1| > |x - 5|$.*

We break the line into three regions:

(1) $x \leq 1$*: In this case, $|x - 1| = 1 - x$ and $|x - 5| = 5 - x$, so*

$$|x - 1| - |x - 5| = (1 - x) - (5 - x) = -4 < 0.$$

This is never positive.

(2) $1 < x < 5$*: In this case, $|x - 1| = x - 1$ and $|x - 5| = 5 - x$, so*

$$\begin{aligned} |x - 1| - |x - 5| &= x - 1 - (5 - x) \\ &= 2x - 6. \end{aligned}$$

This is positive for $3 \leq x < 5$.

(3) $x \geq 5$: In this case, $|x-1| = x-1$ and $|x-5| = x-5$, so

$$|x-1| - |x-5| = x - 1 - (x-5) = 4 > 0.$$

This is always positive.

Putting these together, the solution is:

$$\emptyset \cup [3,5) \cup [5,\infty) = [3,\infty).$$

Example 23 (Simplifying Functions)**.** *Simplify $f(x) = \frac{x^2+2x-35}{x^2+14x+49}$.*

We write

$$\begin{aligned} f(x) &= \frac{(x-5)(x+7)}{(x+7)^2} \\ &= \frac{x-5}{x+7}. \end{aligned}$$

Example 24 (Inverses)**.** *Find the inverse of $f(x) = \frac{x+5}{2x-1}$.*

We solve:

$$y = \frac{x+5}{2x-1}$$

so

$$x + 5 = 2xy - y$$

so

$$x(1-2y) = -y - 5$$

so

$$x = \frac{y+5}{2y-1}.$$

Example 25 (Plotting)**.** *Plot $f(x) = \sqrt{x+5}$ and $g(x) = \cos^2(x)$.*

Example 26 (Lines)**.** *We have $y = mx + b$ for some unknown m, b. You know that $(2, 10)$ and $(5, 12)$ are on the line. Calculate m and b.*