MAT 137Y: Calculus with proofs
Assignment 1 - Sample solutions

Question 1. *In this problem, assume all functions have domain* $\mathbb{R}$*. I will define a new concept. For every pair of functions* f *and* g*, we define the set*

$$\Omega_f^g = \{x \in \mathbb{R} \ : \ f(x) < g(x)\}$$

We say that the function f loves *the function* g *when*

$$\forall x \in \Omega_f^g, \ \exists y \in \Omega_g^f \text{ such that } x < y$$

(a) *Consider the functions* Walt *and* Tor *defined by*

$$\text{Walt}(x) = \sin x, \qquad \text{Tor}(x) = -2\sin x.$$

Prove that Tor *loves* Walt*.*

Suggestion: *Before doing anything else, find out what the sets* $\Omega_{\text{Walt}}^{\text{Tor}}$ *and* $\Omega_{\text{Tor}}^{\text{Walt}}$ *are.*

(b) *Let* $f(x) = 3$ *and let* $g(x) = x$*. Prove that* f *doesn't love* g*.*

(c) *Which functions* f *satisfy that* f *loves* f*?*

Solutions

(a)
- First, I will prove that

$$\Omega_{\text{Walt}}^{\text{Tor}} = \{ x \in \mathbb{R} \mid \exists n \in \mathbb{Z} \text{ s.t. } (2n-1)\pi < x < 2n\pi \} \tag{1}$$

and

$$\Omega_{\text{Tor}}^{\text{Walt}} = \{ x \in \mathbb{R} \mid \exists n \in \mathbb{Z} \text{ s.t. } 2n\pi < x < (2n+1)\pi \} \,. \tag{2}$$

Indeed, by the definitions of Walt and Tor I have that

$$\text{Walt}(x) < \text{Tor}(x) \iff \sin(x) < -2\sin(x) \iff 3\sin(x) < 0\,.$$

Since 3 is a positive real number, the above is equivalent to $\sin(x) < 0$, so Eq. (1) follows directly from the properties of the function sin and the definition of $\Omega_{\text{Walt}}^{\text{Tor}}$. Eq. (2) is obtained in complete analogy.

- Now I want to show that Tor loves Walt; in other words, my goal is to prove that

$$\forall x \in \Omega_{\text{Tor}}^{\text{Walt}}, \ \exists y \in \Omega_{\text{Walt}}^{\text{Tor}} \text{ such that } x < y\,.$$

To prove this, fix $x \in \Omega_{\text{Tor}}^{\text{Walt}}$, and let $y = x + \pi$. Clearly $x < y$. I will now prove that $y \in \Omega_{\text{Walt}}^{\text{Tor}}$.

By Eq. (2), there exists an integer n such that

$$2n\pi < x < (2n+1)\pi\,.$$

I can add π to all three expressions in the chain of inequalities to get

$$(2n+1)\pi < x < (2n+2)\pi$$

or, equivalently

$$(2m-1)\pi < x < 2m\pi$$

with $m = n+1$. This proves that $y \in \Omega^{\text{Tor}}_{\text{Walt}}$.

I have proven that there is indeed an element $y \in \Omega^{\text{Tor}}_{\text{Walt}}$ such that $x < y$. Since x was arbitrary, this proves that Tor loves Walt.

(b) I need to show that the negation of "f loves g" is true. In other words, I need to argue that there exists some $x \in \Omega^{g}_{f}$ for which I cannot find $y \in \Omega^{f}_{g}$ such that $x < y$. Said even differently, I need to prove that

$$\exists x \in \Omega^{g}_{f} \text{ such that } \forall y \in \Omega^{f}_{g},\ x \geqslant y\,. \tag{3}$$

Let $x = \pi$. Then:

- $\pi \in \Omega^{g}_{f}$ because $f(x) = 3 < \pi = g(x)$.
- Moreover, if $y \in \Omega^{f}_{g}$ is fixed then

$$y = g(y) < f(y) = 3 \leqslant \pi = x\,.$$

This shows that $x = \pi$ is an element as in (3), as desired.

(c) Every function f loves itself. Indeed, fix f and notice that

$$\Omega^{f}_{f} = \{\, x \in \mathbb{R} \mid f(x) < f(x) \,\} = \emptyset\,.$$

Therefore, the statement

$$\forall x \in \Omega^{f}_{f},\ \exists y \in \Omega^{f}_{f} \text{ such that } x < y$$

is vacuously true, which proves the assertion.

Question 2. *We continue with the assumptions, notation and definitions as in Question 1. Given a function* f *and any* $t \in \mathbb{R}$*, we define a new function, called* f_t*, via the equation*

$$f_t(x) = f(x) + t.$$

Determine whether each of the following claims is true or false. If true, prove it directly. If false, prove it with a counterexample.

(a) Let f*,* g*, and* h *be functions. IF* f *loves* g *and* g *loves* h*, THEN* f *loves* h*.*

Suggestion: *It may be helpful to think of functions in terms of graphs instead of in terms of their equations at first.*

(b) For every function f *there exists a function* g *such that, for every* $t \in \mathbb{R}$*,* g *loves* f_t*.*

Solutions

(a) The claim is false. I will prove it with a counterexample.
Let $f(x) = -\frac{3}{2}$, $g = -2\sin(x)$, and $h(x) = \sin(x)$.

- We already know from Question 1a that g loves h.
- I will prove that f loves g. For every $n \in \mathbb{Z}$, let us call

$$c_n = \left(2n + \frac{1}{2}\right)\pi$$

  Notice that

$$c_n \in \Omega_g^f$$

  because

$$g(c_n) = -2 < -\frac{3}{2} = f(c_n)$$

  Therefore, if x is *any* real number, there exists an element $y \in \Omega_g^f$ such that $x < y$: just take $y = c_n$ for sufficiently large $n \in \mathbb{Z}$.
  This is true, in particular, for every $x \in \Omega_f^g$. Therefore, f loves g.
- However, f does *not* love h: $\Omega_f^h = \mathbb{R}$ and $\Omega_h^f = \emptyset$, since $f(x) < h(x)$ for every real number x. The statement

$$\forall x \in \mathbb{R},\ \exists y \in \emptyset \text{ such that } x < y$$

  is not true.

This shows that there exist functions f, g, and h such that f loves g and g loves h, but f does not love h.

(b) The claim is true.

Let us fix a function f. I define the function g via the equation

$$g(x) = f(x) + x.$$

Fix $t \in \mathbb{R}$. I will prove that g loves f_t.

For every $x \in \mathbb{R}$ I have that

$$g(x) < f_t(x) \iff f(x) + x < f(x) + t \iff x < t$$

and therefore

$$\Omega_{g}^{f_t} = (-\infty, t)\,,$$

and similarly

$$\Omega_{f_t}^{g} = (t, \infty)\,.$$

Therefore, for every $x \in \Omega_{g}^{f_t}$ the real number $y = t + 1$ is an element of $\Omega_{f_t}^{g}$, and it is bigger than x. I conclude that g loves f_t. Since t was arbitrary I have proved the statement.

Question 3. *Prove by induction that for every positive integer* n*, the number* $5^{2n} + 11$ *is a multiple of 12.*

Proof. The base step corresponds to $n = 1$, in which case I have that $5^{2n} + 11 = 36$, which is a multiple of 12.
For the induction step, let $n \geqslant 1$ be fixed and assume that there exists an integer a such that

$$5^{2n} + 11 = 12a\,.$$

In that case we can write

$$\begin{aligned} 5^{2(n+1)} + 11 &= 5^2 \cdot 5^{2n} + 11 = \\ &= 25 \cdot 5^{2n} + 11 = \\ &= (24 - 1) \cdot 5^{2n} + 11 = \\ &= 24 \cdot 5^{2n} + 5^{2n} + 11 = \\ &= 24 \cdot 5^{2n} + \left(5^{2n} + 11\right) = \qquad \text{(by induction hypothesis)} \\ &= 24 \cdot 5^{2n} + 12a = \\ &= 12 \cdot (2 \cdot 5^{2n}) + 12a = \\ &= 12 \cdot (2 \cdot 5^{2n} + a)\,. \end{aligned}$$

Therefore, $5^{2(n+1)} + 11 = 12b$, where $b = 2 \cdot 5^{2n} + a$ is an integer number.

This shows that if $5^{2n} + 11$ is a multiple of 12 then so is $5^{2(n+1)} + 11$, which is the induction step. This concludes the proof. $\square$