COMP 2080 Winter 2021 Homework Assignment 1

due Wednesday, February 3, 11:59pm

Must be submitted on Crowdmark, **not** UM Learn

Proof questions will be graded for clarity of presentation, as well as correctness.

1. *[2 marks]* There is a mathematical fact that whenever you multiply two odd numbers together, the answer is always odd. Write this fact as a logical formula, and the only allowed symbols are numbers, variables, parentheses, $\neg, \vee, \wedge, \Rightarrow, *, +, =, \in, \mathbb{Z}, \forall, \exists$.
   Hint: by definition, an integer x is odd if it can be written in the form $2k + 1$ for some integer k.

2. Consider the following statement:
   for all $x \in \mathbb{Z}$, if x is written as a product of integers $a \cdot b$, then $a = 1$ or $b = 1$.

   In symbolic form, this can be written as a logical formula S defined by

   $$\forall x \in \mathbb{Z}, \forall a \in \mathbb{Z}, \forall b \in \mathbb{Z}, ((x = a \cdot b) \Rightarrow ((a = 1) \vee (b = 1)))$$

   Define $P(x, a, b)$ to be the predicate $(x = a \cdot b) \Rightarrow ((a = 1) \vee (b = 1))$.

   (a) *[1 mark]* State the converse of the predicate $P(x, a, b)$.

   (b) *[2 marks]* State the contrapositive of the predicate $P(x, a, b)$.
   Simplify (showing steps) so that your final answer does not contain a negation symbol (e.g., $\neg$)

   (c) *[2 marks]* State the inverse of the predicate $P(x, a, b)$.
   Simplify (showing steps) so that your final answer does not contain a negation symbol (e.g., $\neg$)

   (d) *[3 marks]* State the negation of S in symbolic form.
   Simplify (showing steps) so that your final answer does not contain a negation symbol (e.g., $\neg$)

   (e) **[Optional, will not be graded]** Prove or disprove S.

3. *[6 marks]* Consider the sequence $x_1, x_2, \ldots$ defined by:
   $x_1 = 6$, $x_2 = 11$, and $x_i = 3x_{i-1} - 2x_{i-2}$ for all $i > 2$

   Using induction, prove that the following fact is true.

   Fact 1 *For all integers $n \geq 1, x_n = 5 \cdot 2^{n-1} + 1$*

4. *[4 marks]* First, let's recall some basic terminology about triangles. A *right-angled triangle* is a triangle in which one of the angles is exactly 90°. The *hypotenuse* of a right-angled triangle is the side of the triangle that is opposite the 90° angle (i.e., not involved in forming the 90° angle).

   Prove that the following fact is true, using a proof by contradiction.
   (Yes, there are valid proofs that don't use contradiction, but we want you to get practice using this technique. The last step of your proof should reach a contradiction with a well-known fact about right-angled triangles.)

   Fact 2 *Consider any right-angled triangle T with sides of length x, y, h, where h is the length of the hypotenuse and $x, y > 0$. In such a triangle T, we have $x + y > h$.*

5. Consider the following statement S:
   $\exists M \in \mathbb{R}^+, \exists n_0 \in \mathbb{Z}^+, \forall n \in \mathbb{Z}^+, [(n > n_0) \Rightarrow (8 \geq M \cdot \log n)]$

   (a) *[3 marks]* State the negation of S.
   Simplify (showing steps) so that your final answer does not contain a negation symbol (e.g., $\neg$)

   (b) *[6 marks]* Disprove S.
   Clarification: $\mathbb{R}^+$ is the set of all real numbers strictly greater than 0, and $\mathbb{Z}^+$ is the set of all integers strictly greater than 0. The logarithm is with respect to base 2, i.e., writing $\log n$ means $\log_2 n$. You may also use the fact that, for all a, b, we have $(a < b) \Leftrightarrow (\log a < \log b)$