

3. Secants, Tangents, and Limits (Sept. 15)

Lec 2 mini review.

**composition**  $(f \circ g)(x) = f(g(x))$

**inverse**  $f^{-1}(y) = x \iff f(x) = y$

one-to-one function      horizontal line test

**logarithmic functions:** laws of logs

base  $a > 0$        $y = \log_a x$

natural logarithm  $y = \ln x$

inverse relationship between  $\log_a x$  and  $a^x$

**exponential functions:** laws of exponents

base  $a > 0$        $y = a^x$

natural base  $e = 2.718 \dots$        $y = e^x$

**trig ratios:**  $\sin \theta, \cos \theta, \tan \theta, \csc \theta, \sec \theta, \cot \theta$

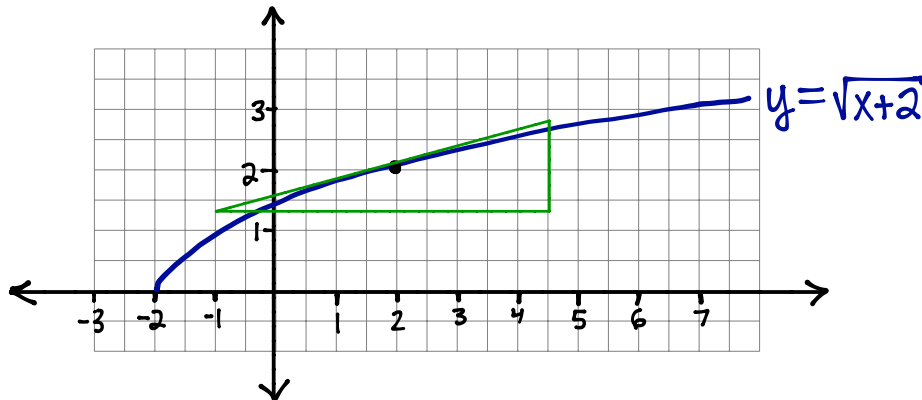
**identities:**  $\cos^2(x) + \sin^2(x) = 1 \dots$  and others

**trig functions:** domain, range

**inverse trig functions:**  $\arcsin x, \arccos x, \arctan x$

GOAL: FIND THE SLOPE OF A TANGENT TO A CURVE

**Example 3.1.** Guess the slope of  $f(x) = \sqrt{x+2}$  at the point  $x = 2$ . Sketch the graph of  $f$  and think about how you can make the best guess possible.



1. guess by "eyeballing" a tangent line and measuring its slope  
mine looks like  $\frac{\Delta y}{\Delta x} = \frac{3}{11} \approx 0.2727 \dots$

2. guess by calculating the slope of a reasonable secant

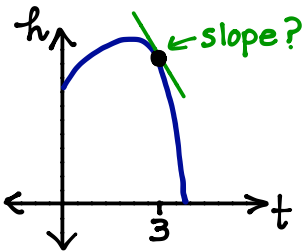
ex  $\left( \begin{array}{l} \text{slope of secant} \\ \text{passing through} \\ (2, f(2)) \text{ and } (3, f(3)) \end{array} \right) = \frac{f(3) - f(2)}{3 - 2} = \frac{\sqrt{5} - \sqrt{4}}{1} \approx 0.236 \dots$

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**Example 3.2.** Recall the height function  $h(t) = -4.9t^2 + v_0t + h_0$  of an object thrown with initial upwards velocity  $v_0$ , from an initial height  $h_0$ ,  $t$  seconds after being thrown.

A ball is thrown up with an initial velocity of 10 m/s from the upper observation deck of the CN Tower, 450 m above the ground. What is the average velocity of the ball over the time interval  $[3, 4]$ ? Estimate the instantaneous velocity of the ball 3 seconds after being thrown. How could you improve your estimate?

$$h(t) = -4.9t^2 + 10t + 450$$



To improve our estimate, let's shrink the interval.

$$\left( \begin{array}{l} \text{average} \\ \text{velocity} \\ \text{on } [3, 3.5] \end{array} \right) = -21.85 \text{ m/s}$$

$$\left( \begin{array}{l} \text{average} \\ \text{velocity} \\ \text{on } [3, 3.1] \end{array} \right) = -19.89 \text{ m/s}$$

$$\left( \begin{array}{l} \text{average} \\ \text{velocity} \\ \text{on } [3, 3.00] \end{array} \right) = -19.4049 \text{ m/s}$$

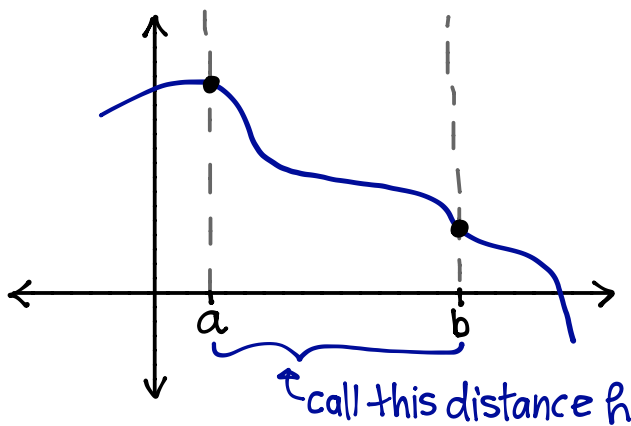
⋮

$$\begin{aligned} \left( \begin{array}{l} \text{average} \\ \text{velocity} \\ \text{on } [3, 4] \end{array} \right) &= \left( \begin{array}{l} \text{slope of secant} \\ \text{through the points} \\ (3, h(3)) \text{ \& } (4, h(4)) \end{array} \right) \\ &= \frac{h(4) - h(3)}{4 - 3} \\ &= -24.3 \text{ m/s} \end{aligned}$$

### SLOPE OF A SECANT — AVERAGE RATE OF CHANGE OVER AN INTERVAL

Let  $f$  be a function that is "continuous" (to be defined precisely later) on an interval  $[a, b]$ .

Then



$$\left( \begin{array}{l} \text{average} \\ \text{rate of change} \\ \text{of } f \text{ on } [a, b] \end{array} \right) = \left( \begin{array}{l} \text{slope of secant} \\ \text{through the points} \\ (a, f(a)) \text{ and } (b, f(b)) \end{array} \right)$$

$$= \frac{f(b) - f(a)}{b - a}$$

$$= \frac{f(a+h) - f(a)}{h}$$

$$\begin{array}{c} a \quad \quad \quad b = a + h \\ \underbrace{\hspace{1.5cm}} \\ h \end{array}$$

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## SLOPE OF TANGENT — INSTANTANEOUS RATE OF CHANGE AT A POINT

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**Goal:** We want the **instantaneous rate of change** of  $f(x)$  at a point  $x = a$ .

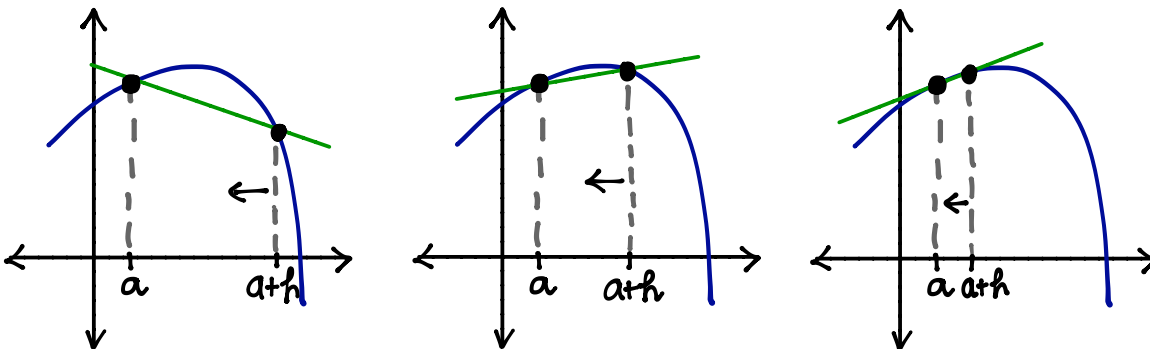
In this case, the “interval” we are interested in is  $[a, a]$ . That is, we only care what happens when  $x$  is exactly equal to  $a$ .

**Obstacle:** The formula for the average rate of change of  $f$  on the interval  $[a, a]$  does not work — we get the **indeterminate form**  $\frac{0}{0}$ .

$$\left( \begin{array}{l} \text{average rate of change} \\ \text{of } f \text{ on the "interval"} \\ [a, a] \end{array} \right) = \frac{f(a) - f(a)}{a - a} = \frac{0}{0} \quad \text{☹️}$$

**Observation:** If  $h > 0$ , then we can calculate the average rate of change over the interval  $[a, a + h]$ , even when  $h$  is extremely tiny.

$$\left( \begin{array}{l} \text{average rate of change} \\ \text{of } f \text{ on the interval} \\ [a, a+h] \end{array} \right) = \frac{f(a+h) - f(a)}{a+h-a} = \frac{f(a+h) - f(a)}{h} \quad \leftarrow \text{denom. } \neq 0 \quad \text{😊}$$



Idea Keep shrinking  $h$ ...  
we want  $h$  to approach 0  
written  $h \rightarrow 0$

So,  $h$  can **approach** 0, written  $h \rightarrow 0$ , without ever actually equalling zero. At the same time, if the average rate of change of  $f$  over the interval  $[a, a + h]$  **approaches** a particular number, then that number is called the **instantaneous rate of change of  $f$  at  $x = a$** .

We need to formalize the idea of  $h \rightarrow 0$ . In fact, we will develop a framework for evaluating limits in general, not just those for instantaneous rates of change.

## LIMITS: THE INTUITIVE DEFINITION

Suppose  $f(x)$  is defined when  $x$  is “near” a number  $a$  (this means that  $f$  is defined on some open interval that contains the number  $a$ , except possibly at  $a$  itself;  $a$  might not be in the domain of  $f$ , but at all other points in the neighbourhood of this open interval,  $f$  is defined).

- ▶ If we can make the values  $f(x)$  arbitrarily close to a unique real number  $L$  by restricting  $x$  (on either side of  $a$ ) to be sufficiently close to  $a$  but not equal to  $a$ , then

$$\lim_{x \rightarrow a} f(x) = L$$

[read: “the limit of  $f(x)$ , as  $x$  approaches  $a$ , exists and equals  $L$ ”]

**Informally**, we can guarantee that  $f(x)$  gets arbitrarily close to a unique real number  $L$  as long as we make sure that  $x$  is close enough to  $a$  (without actually letting  $x$  equal  $a$ ).

- ▶ If there is no such unique real number  $L$ , then the limit of  $f(x)$  as  $x$  approaches  $a$  **does not exist (DNE)**.

**Example 3.3.** Consider the rational function  $f(x) = \frac{2x^2 - 2x}{x - 1}$  and the limit  $\lim_{x \rightarrow 1} f(x)$ .

- ◊ What happens if we just plug in  $x = 1$  to  $f(x)$ ?

$f(1)$  is undefined since denominator would be zero.

- ◊ Test how  $f(x)$  behaves for values of  $x$  near  $x = 1$  by filling in the chart:

(from the left  $x \rightarrow 1$ )      ( $1 \leftarrow x$  from the right)

$x$	0.5	0.75	0.9	0.99	0.999	1	1.001	1.01	1.1	1.25	1.5
$f(x)$	1	1.5	1.8	1.98	1.998	$\frac{0}{0}$ eek!	2.002	2.02	2.2	2.5	3

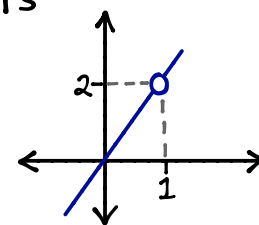
- ◊ As  $x$  approaches 1, does  $f(x)$  seem to be approaching a specific number?

Yes, it looks like  $f(x) \rightarrow 2$

- ◊ If  $x$  is any number other than 1, what does the graph of  $f$  look like?

$f(x) = \frac{2x(x-1)}{(x-1)}$  ← if  $x \neq 1$ , then these are nonzero factors which are equal and can be cancelled.

everywhere except at  $x=1$ ,  $f(x)$  looks like the line  $y=2x$



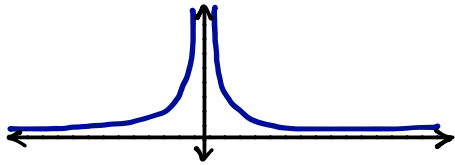
- ◊ Use the graph of  $f$  to evaluate the limit:  $\lim_{x \rightarrow 1} f(x) = 2$

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## REASONS WHY SOME LIMITS DO NOT EXIST

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Example 3.4.  $\lim_{x \rightarrow 0} \frac{1}{x^2}$  DNE



- as  $x \rightarrow 0$ ,  $\frac{1}{x^2}$  keeps growing larger
- $\Rightarrow \frac{1}{x^2}$  does not approach a unique real #  $L$
- $\therefore$  limit DNE

### Infinite Limits (Vertical Asymptotes)

- Let  $f$  be a function defined on both sides of  $a$ , except possibly at  $a$  itself. Then

$$\lim_{x \rightarrow a} f(x) = \infty$$

means that the values of  $f(x)$  grow arbitrarily large as  $x$  approaches  $a$ .

- **Graphically:**  $f$  has a **Vertical Asymptote** as  $x$  approaches  $a$ .

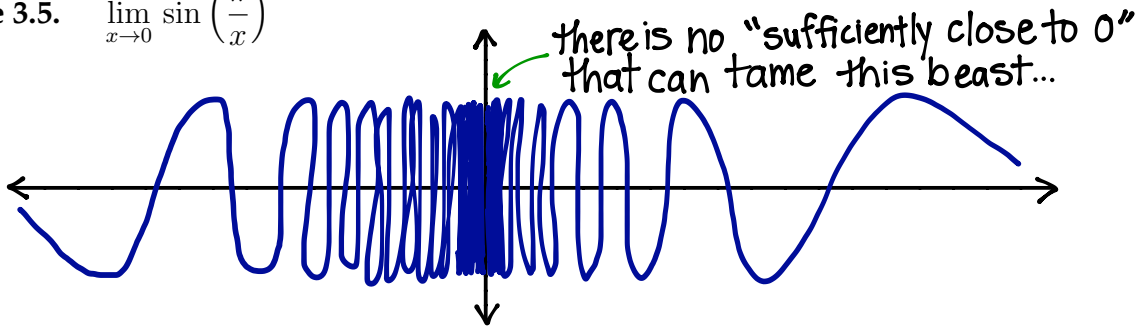
- Same idea for  $\lim_{x \rightarrow a} f(x) = -\infty$

- **Note.** Since  $\infty$  is **not** a real number  $L \in \mathbb{R}$ , infinite limits DNE.

Nevertheless, we write  $\lim_{x \rightarrow a} f(x) = \infty$  or  $\lim_{x \rightarrow a} f(x) = -\infty$

because it tells us for short which way the Vertical Asymptote goes.

Example 3.5.  $\lim_{x \rightarrow 0} \sin\left(\frac{\pi}{x}\right)$



**Observations.** As  $x \rightarrow 0$ , it happens **infinitely often** that

- $\sin\left(\frac{\pi}{x}\right) = 0$  For  $k = 1, 2, 3, \dots$  using  $x = \frac{1}{k}$  gives  $\sin\left(\frac{\pi}{x}\right) = \sin(k\pi) = 0$   
and  $\frac{1}{k} = 1, \frac{1}{2}, \frac{1}{3}, \dots \rightarrow 0$

- $\sin\left(\frac{\pi}{x}\right) = 1$  For  $k = 1, 2, 3, \dots$  using  $x = \frac{2}{4k+1}$  gives  $\sin\left(\frac{\pi}{x}\right) = \sin((4k+1)\frac{\pi}{2}) = 1$   
and  $\frac{1}{k} = 1, \frac{1}{2}, \frac{1}{3}, \dots \rightarrow 0$
- same idea...

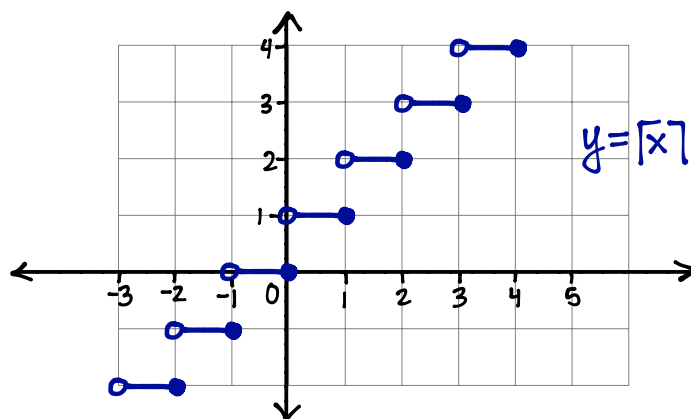
- Since  $\sin\left(\frac{\pi}{x}\right)$  does not approach a **unique** real number, as  $x \rightarrow 0$ , this limit DNE.

**Example 3.6.** For all  $x \in \mathbb{R}$ , the ceiling function  $\lceil x \rceil$  is defined as  $\lceil x \rceil = \min\{n \in \mathbb{Z} : x \leq n\}$ .

$$\lim_{x \rightarrow 2} \lceil x \rceil \quad \text{DNE}$$

from left  
 $x \rightarrow 2$  and  $x < 2$   
 $\lceil x \rceil = 2$

from right  
 $x \rightarrow 2$  and  $x > 2$   
 $\lceil x \rceil = 3$



► Since  $\lceil x \rceil$  approaches **different real numbers** as  $x \rightarrow 2$  from either side, this limit DNE.

### ONE-SIDED LIMITS

- ◇ As in Example 3.6, as  $x \rightarrow a$ , the values of  $f(x)$  may behave differently from one side than the other.
- ◇ For some functions, a limit as  $x \rightarrow a$  only makes sense if  $x$  approaches  $a$  from one side:

Ex.  $\lim_{x \rightarrow 0} \sqrt{x}$  only makes sense if  $x > 0$

To distinguish from which side  $x$  approaches  $a$ , we use the following notation for **one-sided limits**:

left-side limit

$$\lim_{x \rightarrow a^-} f(x)$$

↑  
 superscript  
 minus sign  
 means  $x \rightarrow a$   
 and  $x < a$

right-side limit

$$\lim_{x \rightarrow a^+} f(x)$$

↑  
 superscript  
 plus sign  
 means  $x \rightarrow a$   
 and  $x > a$

By definition, we can say

$$\lim_{x \rightarrow a} f(x) = L \quad \text{if and only if both} \quad \lim_{x \rightarrow a^-} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow a^+} f(x) = L$$

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## EVALUATING LIMITS

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- ▶ **numerically:** guessing by plugging in nearby values of  $x$
- ▶ **graphically:** eyeballing the limit by looking at the graph
- ▶ **using the Limit Laws:**

Let  $k$  be a constant real number, and suppose that the limits  $\lim_{x \rightarrow a} f(x)$  and  $\lim_{x \rightarrow a} g(x)$  both exist. Then

$$\lim_{x \rightarrow a} [f(x) \pm g(x)] = \left[ \lim_{x \rightarrow a} f(x) \right] \pm \left[ \lim_{x \rightarrow a} g(x) \right]$$

$$\lim_{x \rightarrow a} [kf(x)] = k \left[ \lim_{x \rightarrow a} f(x) \right]$$

$$\lim_{x \rightarrow a} [f(x)g(x)] = \left[ \lim_{x \rightarrow a} f(x) \right] \cdot \left[ \lim_{x \rightarrow a} g(x) \right]$$

$$\lim_{x \rightarrow a} \left[ \frac{f(x)}{g(x)} \right] = \frac{\left[ \lim_{x \rightarrow a} f(x) \right]}{\left[ \lim_{x \rightarrow a} g(x) \right]} \quad \text{if } \lim_{x \rightarrow a} g(x) \neq 0.$$

- ▶ **using direct substitution:**

If  $f$  is a polynomial or a rational function and  $a$  is in the domain of  $f$ , then

$$\boxed{\lim_{x \rightarrow a} f(x) = f(a)}.$$

- ▶ **using algebraic tricks:**

If  $f(x) = g(x)$  everywhere except when  $x = a$ , then

$$\boxed{\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)} \quad \text{provided the limit exists.}$$

The above observation allows us to use **algebraic tricks** (such as the following) to evaluate limits:

- ♣ factoring and cancelling common factors
- ♡ rationalizing the numerator or denominator
- ♠ dividing all terms by a common expression
- ◇ adding/subtracting fractional expressions on a common denominator

Example 3.7.  $\lim_{t \rightarrow 0} \frac{\sqrt{t^2+9} - 3}{t^2}$  num.  $\rightsquigarrow 0$  denom.  $\rightsquigarrow 0$  indeterminate  $\frac{0}{0}$

$$= \lim_{t \rightarrow 0} \left( \frac{\sqrt{t^2+9} - 3}{t^2} \right) \left( \frac{\sqrt{t^2+9} + 3}{\sqrt{t^2+9} + 3} \right) \quad (\text{rationalize numerator})$$

$$= \lim_{t \rightarrow 0} \frac{(\sqrt{t^2+9})^2 + 3\sqrt{t^2+9} - 3\sqrt{t^2+9} - 9}{t^2(\sqrt{t^2+9} + 3)}$$

$$= \lim_{t \rightarrow 0} \frac{t^2 + 9 - 9}{t^2(\sqrt{t^2+9} + 3)}$$

$$= \lim_{t \rightarrow 0} \frac{t^2}{t^2(\sqrt{t^2+9} + 3)} \quad \leftarrow \text{everywhere except } t=0, \frac{t^2}{t^2} \text{ cancels}$$

$$= \lim_{t \rightarrow 0} \frac{1}{\sqrt{t^2+9} + 3}$$

$$= \frac{1}{\sqrt{9+3}} = \frac{1}{6}$$

$$\therefore \lim_{t \rightarrow 0} \frac{\sqrt{t^2+9} - 3}{t^2} = \frac{1}{6}$$

Example 3.8.  $\lim_{x \rightarrow 1} \left( \frac{1}{x-1} - \frac{1}{x^2-x} \right)$

$$= \lim_{x \rightarrow 1} \left( \frac{1}{x-1} - \frac{1}{x(x-1)} \right)$$

$$= \lim_{x \rightarrow 1} \left( \frac{x}{x(x-1)} - \frac{1}{x(x-1)} \right)$$

$$= \lim_{x \rightarrow 1} \left( \frac{x-1}{x(x-1)} \right) \quad \leftarrow \text{everywhere except } x=1, (x-1)/(x-1) \text{ cancels}$$

$$= \lim_{x \rightarrow 1} \frac{1}{x}$$

$$= 1$$

$$\therefore \lim_{x \rightarrow 1} \left( \frac{1}{x-1} - \frac{1}{x(x-1)} \right) = 1$$

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## LIMITS: THE PRECISE DEFINITION

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In the Intuitive Definition for Limits, the wording “arbitrarily close” and the promised guarantee if  $x$  is “sufficiently close” sounds a bit vague... it is.

Here is the Precise Definition for Limits:

Let  $f$  be a function defined on some open interval that contains the number  $a$ , although  $f$  need not be defined at  $a$  itself. Then what is precisely meant when we say

$$\lim_{x \rightarrow a} f(x) = L$$

is the following: For every number  $\varepsilon > 0$  number, there exists a number  $\delta > 0$  such that

within  $\delta$  of  $a$  is  
"sufficiently close"

$$0 < |x - a| < \delta \quad \text{guarantees that} \quad |f(x) - L| < \varepsilon.$$

within  $\varepsilon$  of  $L$  is  
"arbitrarily close"

**Remark.** The Intuitive Definition of  $\lim_{x \rightarrow a} f(x)$  existing and equalling  $L$  is an (arguably) over-simplified interpretation of the formal definition:

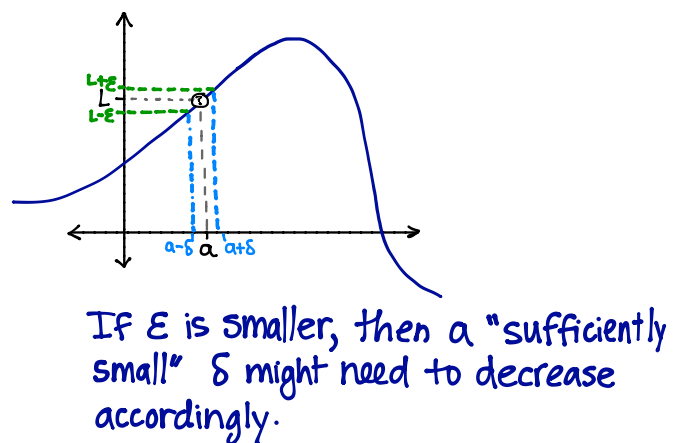
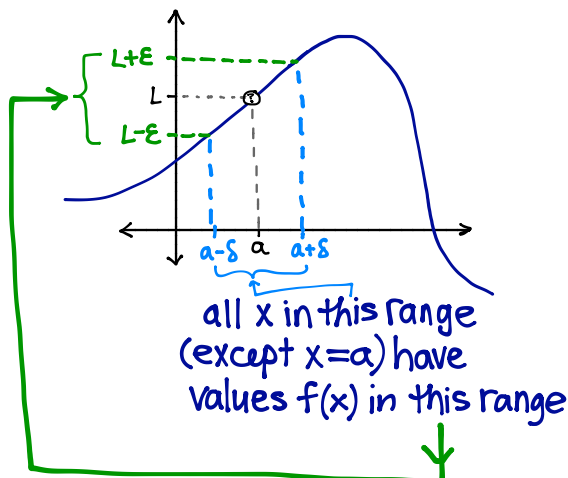
No matter how tiny (arbitrarily small)  $\varepsilon$  is (without actually being zero), we can make sure that  $f(x)$  is no further than  $\varepsilon$  away from  $L$ ; we just need to carefully pick the “right”  $\delta > 0$  to do the job for the given  $\varepsilon$ .

### The $\varepsilon$ - $\delta$ Game

So how do we make use of the Precise Definition for Limits?

▷ We imagine we are playing this game (hooray!):

- ◊ I have chosen  $\varepsilon > 0$ . Maybe  $\varepsilon = 0.1$ . Maybe  $\varepsilon = 10^{-99}$ . I don't tell you what  $\varepsilon$  is explicitly; all you know for sure it that  $\varepsilon > 0$ .
- ◊ Now, you must provide me with a number  $\delta > 0$  that guarantees that, other than  $x = a$  itself, all  $x$ 's in the range  $(a - \delta, a + \delta)$  will have values  $f(x)$  in my chosen range  $(L - \varepsilon, L + \varepsilon)$ .
- ◊ Question to ponder: How can you express your answer for  $\delta$  without explicitly knowing how large/tiny my  $\varepsilon$  is?



**Exercise 3.9.** Consider the rational function  $f(x) = \frac{2x^2 - 2x}{x - 1}$ .

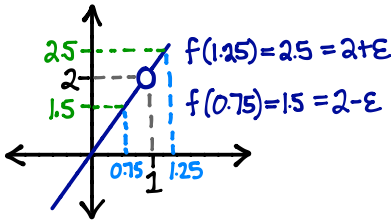
Prove that  $\lim_{x \rightarrow 1} f(x) = 2$  using the precise definition of limits.

Start by considering a few specific values for  $\varepsilon > 0$ ; for each give the corresponding  $\delta > 0$  that makes

$$0 < |x - 1| < \delta \quad \text{guarantee that} \quad |f(x) - 2| < \varepsilon.$$

Then find a general answer for  $\delta$  that depends on the given  $\varepsilon$  (without needing to know what number  $\varepsilon$  represents exactly).

Ex. Say  $\varepsilon = 0.5$



From the picture, it looks like  $\delta = 0.25$  would be sufficiently small for  $\varepsilon = 0.5$

proof. Suppose  $|x - 1| < 0.25$  and  $x \neq 1$

$$\text{Then } 2|x - 1| < 2(0.25)$$

$$\Rightarrow 2|x - 1| < 0.5 = \varepsilon$$

$$\Rightarrow |2(x - 1)| < 0.5 = \varepsilon \quad (\text{since } 2 > 0)$$

$$\Rightarrow |2x - 2| < 0.5 = \varepsilon$$

$$\Rightarrow \left| \frac{2x(x-1)}{(x-1)} - 2 \right| < 0.5 = \varepsilon \quad (\text{since } x \neq 1)$$

$$\Rightarrow |f(x) - 2| < 0.5 = \varepsilon$$



For arbitrarily small  $\varepsilon > 0$ , we claim that  $\delta = \varepsilon/2$  is sufficiently small.

• since  $\varepsilon > 0$ , it is also true that  $\frac{\varepsilon}{2} > 0$

•  $\therefore \delta = \varepsilon/2 > 0$ , which is required by definition.

Now, suppose  $|x - 1| < \delta$  and  $x \neq 1$

$$\text{Then } 2|x - 1| < 2\delta$$

$$\Rightarrow 2|x - 1| < 2\left(\frac{\varepsilon}{2}\right) = \varepsilon \quad (\text{since we chose } \delta = \frac{\varepsilon}{2})$$

$$\Rightarrow |2(x - 1)| < \varepsilon \quad (\text{since } 2 > 0)$$

$$\Rightarrow |2x - 2| < \varepsilon$$

$$\Rightarrow \left| \frac{2x(x-1)}{(x-1)} - 2 \right| < \varepsilon \quad (\text{since } x \neq 1)$$

$$\Rightarrow |f(x) - 2| < \varepsilon$$

No matter what  $\varepsilon > 0$  was chosen, taking  $\delta = \varepsilon/2$  is good enough for this function.

[for other functions,  $\delta$  might be more complicated...]

## STUDY GUIDE

### Important terms and concepts:

- ◇ slope of a secant      average rate of change of a function
- ◇ goal: find slope of a tangent      instantaneous rate of change of a function
- ◇ **limits and one-sided limits:**       $\lim_{x \rightarrow a} f(x)$        $\lim_{x \rightarrow a^+} f(x)$        $\lim_{x \rightarrow a^-} f(x)$
- ◇ **why some limits DNE:** infinite, no unique  $L$ , different from left/right
- ◇ **evaluating limits:** numerically, graphically, with Limit Laws and algebraic tricks

Exercises: Stewart, 8th ed.

§2.1, pg. 82 # 3, 5

§2.2 pg. 92 # 5, 7, 11, 23, 31, 33, 41

§2.3 pg. 102 # 1, 11, 13, 19, 21, 23, 25, 27, 31, 41

optional exercises: §2.4 pg. 114 # 23, 24, 25