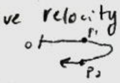
Dynamics - MIE100 - Study Notes

Useful Conversions

- $1\frac{rev}{min}\left(\frac{1min}{60s}\right)\left(\frac{2\pi}{1rev}\right) = \_\ rad/s$ ↳ $\dot{\theta}$
- 1 inch $= \_$ ft ($\div 12$)
- 1 km/h $= \_$ m/s ($\div 3.6$)
- 1 mi/h $= \_$ ft/s ($\times 1.467$)
- rad → deg $\left(\frac{180}{\pi}\right)$
- deg → rad $\left(\frac{\pi}{180}\right)$
- mass $= \frac{w}{g}\left[\frac{\_\ lb}{32.2\ ft/s^2}\right]$

2.2 - Rectilinear Motion

- Kinematics - study of motion
- Kinetics - study of forces and resultant motion
- Ave velocity vs speed: $\boxed{\vec{V}_{ave} = \frac{\Delta s}{\Delta t}}$ $\boxed{V_{speed\ ave} = \frac{\Delta S_T}{\Delta t}}$ → total distance travelled (path independent) ↳ $S_T = |s(t_2) - s(t_1)| + |s(t_2) - s(t_3)|$



- Equations of motion
  - Velocity: $\boxed{v = \frac{ds}{dt}}$
  - Acceleration: $\boxed{a = \frac{dv}{dt} = v\frac{dv}{ds}}$ ⇒ $\boxed{a\,ds = v\,dv}$

$* a = a(s) \to a(s) = \frac{dv}{dt} \} dt = \frac{ds}{v} = \frac{dv}{a}$

- $\boxed{s(t) = s_0 + v_0 t}$ ~ $a = 0$
- $\boxed{v(t) = v_0 + a_0 t}$ → constant a; $\boxed{s(t) = s_0 + v_0 t + \frac{1}{2}a_0 t^2}$
- $\boxed{v = v_0 + \int a(t)\,dt}$ ~ $a = f(t)$; $\boxed{s = s_0 + v_0 t + \int_0^t\left(\int_0^t a(t)\,dt\right)dt}$
- $\boxed{v^2 = v_0^2 + 2a_0\Delta s}$ ~ $a = f(s)$
- $\boxed{t = \int \frac{1}{a(v)}\,dv}$ - $a = f(v)$
- $\boxed{\int v\,dv = \int a(\theta)\,r\,d\theta}$

$* a = 0: 0 = \frac{dv}{dt} \to v = v_0$ ~ $v = \frac{ds}{dt} \sim \int v_0\,dt = \int ds \to v_0 t = s - s_0$

$* a = a_0 \to a_0 = \frac{dv}{dt} \sim \int a_0\,dt = \int dv \to a_0 t = v - v_0$

$\to v(t) = \frac{ds}{dt} = v_0 + a_0 t \to \int ds = \int(v_0 + a_0 t)dt \to s - s_0 = v_0 t + a_0\frac{t^2}{2}$

$* a = a(t) \to a(t) = \frac{dv}{dt} \to \int a(t)\,dt = \int dv \to v(t) = v_0 + \int a(t)dt = \frac{ds}{dt}$

$* a = a(s) \to a(s) = \frac{dv}{dt} \to dt = \frac{dv}{v} = \frac{dv}{a}$

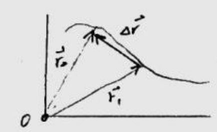
$\boxed{a\,ds = v\,dv} \to \int_{s_0}^{s} a\,ds = \int_{v_0}^{v} v\,dv \to v^2(s) = v_0^2 + 2\int_{s_0}^{s} a(s)\,ds$

$* a = a(v) \to a(v) = \frac{dv}{dt} \to \int dt = \int\frac{dv}{a(v)} \to t = t_0 + \int\frac{1}{a(v)}dv$

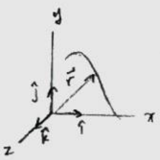
$* a = a(\theta) \to a(\theta) = \frac{dv}{dt}$

2.3 - Plane Curvilinear Motion

- Use FIXED reference 'O'
- Position vector: $\vec{r}(t)$
- Displacement: $\boxed{\Delta\vec{r} = \vec{r}_2 - \vec{r}_1}$
- Velocity: $\boxed{\vec{v} = \frac{d\vec{r}}{dt} = \dot{\vec{r}}}$ ~ speed: $|\vec{v}|$
- acceleration: $\boxed{\vec{a} = \frac{d\vec{v}}{dt} = \dot{\vec{v}}}$



2.4 - Rectangular Coordinates (x-y)



Reminder: $\boxed{\tan\theta = \frac{v_y}{v_x}}$

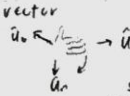
- position: $\boxed{\vec{r} = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}} = \boxed{x\hat{\imath} + y\hat{\jmath} + z\hat{k}} = \boxed{r_x\hat{\imath} + r_y\hat{\jmath} + r_z\hat{k}}$
- velocity: $\boxed{\vec{v} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k}} = \boxed{\dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + \dot{z}\hat{k}} = \boxed{v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k}}$
- acceleration: $\boxed{\vec{a} = \frac{d^2x}{dt^2}\hat{\imath} + \frac{d^2y}{dt^2}\hat{\jmath} + \frac{d^2z}{dt^2}\hat{k}} = \boxed{\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} + \ddot{z}\hat{k}} = \boxed{a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}}$
- Velocity components: $\boxed{|\vec{v}| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} \Rightarrow \boxed{|\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}}$
- Acceleration components: $\boxed{|\vec{a}| = \sqrt{\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2}} \Rightarrow \boxed{|\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}}$

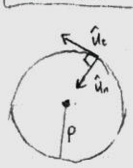
Projectile Motion

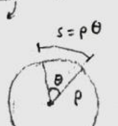
x-axis	y-axis
$\boxed{V_{x0} = V_0\cos\theta}$	$\boxed{V_{y0} = V_0\sin\theta}$
$\boxed{x = x_0 + V_{x0}t}$	$y = y_0 + V_{y0}t + \frac{1}{2}a_{y0}t^2$
$\boxed{a = 0}$	$\boxed{a_y = a_{y0}}$
	$g = -9.8\ m/s^2 = -32.2\ ft/s$
$\boxed{V_x = \dot{x}_0} + gt$, $\boxed{a = 0}$	$\boxed{V_y = V_{y0} + gt}$
	$\boxed{V_y^2 = V_{y0}^2 + 2g(y - y_0)}$

2.5 - Normal / Tangential Coordinates (n-t)

- $\hat{u}_t$: unit vector TANGENT to path
- $\hat{u}_n$: unit vector PERPENDICULAR to $\hat{u}_t$ ↳ concave positive
- $\hat{u}_b$: unit binomial vector $\boxed{\vec{u}_b = \vec{u}_t \times \vec{u}_n}$







Tangential (Δ speed)	Normal (Δ direction)
$\dot{v} \leftarrow \boxed{a_t = \frac{dv}{dt}}$	$\boxed{a_n = v\dot{\theta}}$
$\boxed{s(t) = \rho\theta\hat{u}_t}$	$\boxed{a_n = \frac{v^2}{\rho}}$
$\boxed{\vec{v}(t) = v(t)\hat{u}_t}$	$\boxed{a_n = \rho\dot{\theta}^2}$
$\boxed{\vec{v}(t) = \rho\dot{\theta}\,\hat{u}_t}$	$\boxed{s = \rho\theta}$ $\boxed{\dot{s} = \dot{\rho}\theta}$ $\frac{v^2}{\rho}$

$\boxed{\vec{a}(t) = a_t\hat{u}_t + a_n\hat{u}_n = \dot{v}\hat{u}_t + v\dot{\theta}\hat{u}_n}$

- radius - can find ρ if path described by rectangular function $y = f(x)$

$$\boxed{\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}}$$

- magnitude of a

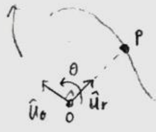
$$\boxed{\vec{a} = \sqrt{a_n^2 + a_t^2}}$$

2.6 - Polar Coordinates (r-θ)

$* \boxed{\omega = \dot{\theta}}$ $\boxed{\dot{\omega} = \ddot{\theta}}$

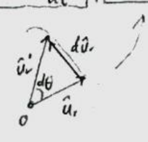
$*$ Units:
- $\theta = \_^\circ$
- $\dot{\theta} = rad/s$
- $\ddot{\theta} = rad/s^2$

- $\hat{u}_r$: unit vector parallel to $\vec{r}$ (coincident)
- $\hat{u}_\theta$: unit vector ⊥ to $\hat{u}_r$
- $\hat{u}_z$: unit vector in z-direction (3D)
- Derive $\frac{d\hat{u}_r}{dt}$:



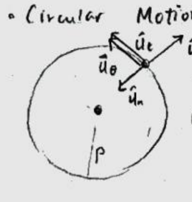
$\boxed{\frac{d\hat{u}_r}{dt} = \frac{d\theta}{dt}\hat{u}_\theta}$ $\boxed{\dot{\hat{u}}_r = \dot{\theta}\hat{u}_\theta}$

$\boxed{\frac{d\hat{u}_\theta}{dt} = \frac{-d\theta}{dt}\hat{u}_r}$ $\boxed{\dot{\hat{u}}_\theta = -\dot{\theta}\hat{u}_r}$



- position vector: ↗ $\vec{r}$ = distance O → P $\boxed{\vec{r} = r\vec{u}_r}$
- velocity vector: $\boxed{\vec{v} = \dot{r}\hat{u}_r + r\dot{\theta}\hat{u}_\theta}$ $* \boxed{V_r = \dot{r}}$ $* \boxed{V_\theta = r\dot{\theta}}$ ↳ $\boxed{|\vec{v}| = \sqrt{V_r^2 + V_\theta^2}}$
- acceleration: $\vec{a} = a_r\hat{u}_r + a_\theta\hat{u}_\theta$
  - $\boxed{a_r = \ddot{r} - r\dot{\theta}^2}$
  - $\boxed{a_\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta}}$
  - $\boxed{|\vec{a}| = \sqrt{a_r^2 + a_\theta^2}}$

Circular Motion

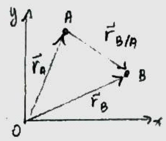


Note: $\boxed{\hat{u}_r = -\hat{u}_n}$ $\boxed{\hat{u}_\theta = \hat{u}_t}$

r-θ ($\rho \neq$ change): $\boxed{a_\theta = r\ddot{\theta}}$ $\boxed{V_\theta = r\dot{\theta}}$ ω; $\boxed{a_r = -r\dot{\theta}^2}$ $\boxed{V_r = 0}$

n-t: $\boxed{a_t = \dot{v} = v\ddot{\theta}}$; $\boxed{a_n = \frac{v^2}{\rho} = r\dot{\theta}^2 = v\dot{\theta}}$

2.8 - Relative Motion

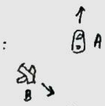


$\vec r_{B/A}$ = position vector of B relative to A

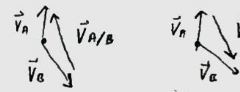
$\boxed{\vec r_B = \vec r_A + \vec r_{B/A}}$ * $\vec r_{B/A}$ ~ graphically from A→B

$\boxed{\vec V_B = \vec V_A + \vec V_{B/A}}$
$\boxed{\vec a_B = \vec a_A + \vec a_{B/A}}$ } rectangular coordinates

Ex:



$V_{A/B}$ (in plane) vs $V_{B/A}$ (in car)

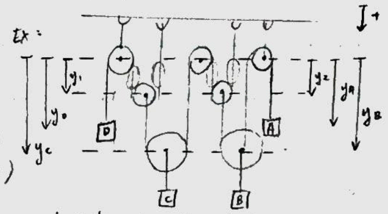


2.9 - Constrained Motion of Connected Particles

* Pay attention to (+ve) & (-ve)

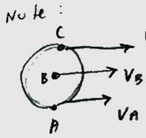
- Pulley Problems
  - Method: (fixed cable)
    1) Count ropes
    2) Draw datum line
    3) identify constants
    4) find y_A, y_B (for each mass)
    5) find y_1, y_2 (different/each rope)
    6) Sum ropes (1 eq/rope)
    7) Differentiate → constants = 0
    8) $L' = 0$ { ⋮ Solve

Ex:



$L_1 = y_D + 2y_1 + C_1 \Rightarrow 0 = V_D + 2V_1$
$L_2 = (y_C - y_1) + y_C + 2y_2 + C_2 \Rightarrow 0 = 2V_C - V_1 + 2V_2$
$L_3 = (y_B - y_2) + y_B + y_A + C_3 \Rightarrow 0 = 2V_B - V_2 + V_A$ } Solve

Note:



Wheel
$V_B = \frac{V_A + V_C}{2}$ average

3.2 - Newton's Second Law / 3.2 - Equation of Motion & Solution of Problem / 3.4 Rectilinear Motion / 3.5 Curvilinear Motion

↳ $\boxed{F = ma}$ → a measured w/ respect inertial/newtonian/non-rotating/non-accelerating FOR.

* $g = -9.81\ m/s^2$
$g = -32.2\ ft/s^2$
* $m = \frac{W}{g} = \left[\frac{lb}{32.2\ ft/s^2}\right]$

- Forces of interest:
  1) Gravity (F_g)

$\boxed{F_g = \frac{G m_1 m_2}{r^2}}$ * $G = 66.73 \times 10^{-12}\ m^3/kg \cdot s^2$
* r = distance betw. m_1 & m_2

$\boxed{F_g = mg}$ $-g = 9.81\ N/kg$

Note: mass ⇒ kg
weight $W = mg$ [N]

  2) Normal Force (F_N)
     - always perp. to surface

  3) Friction (F_f)
     a) Dynamic/Kinetic Friction (oppose motion)

$\boxed{F_f = \mu_K F_N}$ * F_f = friction force [N]
* μ_K = coefficient of kinetic fric.

     b) Static friction (no motion) * $V = 0,\ a = 0$

$\boxed{F_{fs\,max} = \mu_s F_N}$ = F_A (same direction, opp. sense)

$(F_{fs} \le \mu_s F_N)$ * F_s = static friction
* μ_s = coefficient static friction

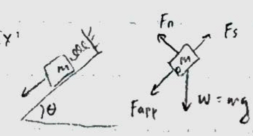
Note: $\mu_s \ne \mu_K$ ~ $\mu_s > \mu_K$ ⟵

  4) Spring/Elastic (F_s)

$\boxed{F_s = -kx = -k\,l_2 - l_1}$ * F_s = spring force
* k = spring constant [N/m]

want restore original (-ve give direction)

Ex:



$\dot\theta$ /w
* deg/s $\left(\frac{\pi}{180}\right)$ = rad/s

- Equations of motion
  - Rectangular

$\sum \vec F = m\vec a$

$\boxed{\begin{aligned} x &: \sum F_x = ma_x \\ y &: \sum F_y = ma_y \\ z &: \sum F_z = ma_z \end{aligned}}$

$\boxed{\vec a = a_x\hat\imath + a_y\hat\jmath + a_z\hat k}$

  - Normal + Tangential

$\sum \vec F = m\vec a$

$\boxed{n : \sum F_n = ma_n = m\frac{v^2}{\rho}}$
$\boxed{t : \sum F_t = ma_t = m\dot v_t}$
$\boxed{b : \sum F_b = 0}$

  - Polar Coordinates

$\sum \vec F = m\vec a$

$\boxed{r : \sum F_r = ma_r = m(\ddot r - r\dot\theta^2)}$
$\boxed{\theta : \sum F_\theta = ma_\theta = m(2\dot r\dot\theta + r\ddot\theta)}$
$\boxed{z : \sum F_z = ma_z = m\ddot z}$

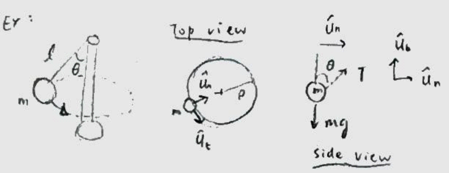
- Equations of motion (System of particles)
  - G = centre of mass

$\boxed{\sum \vec F = m\vec a_G}$

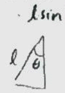
* Centre of mass of system ($\vec r_G$)

$\boxed{\vec r_G = \frac{\sum m_i \vec r_i}{m}}$

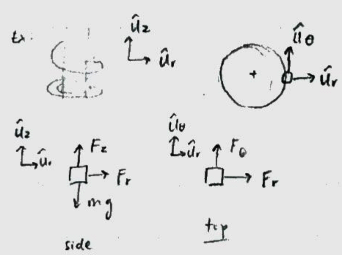
Ex:



$\frac{mv^2}{\rho} = \frac{mv^2}{l\sin\theta} = T\sin\theta = \frac{mg}{\cos\theta}\sin\theta$

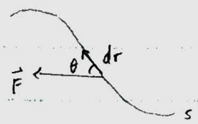


Ex:



3.6 - Work and Kinetic Energy

- work (U) - done by force $\vec{F}$ as it moves over small displacement $d\vec{r}$.



$dU = \vec{F} \cdot d\vec{r} = |\vec{F}||d\vec{r}|\cos\theta = F\cos\theta \, dr$

$\boxed{U = \int_{s_1}^{s_2} F\cos\theta \, ds}$

work $[J] = [N \cdot m]$

- Note:
  - → work = scalar
  - → +ve work ~ force + displacement SAME direction
  - → -ve work ~ force + displacement OPP direction

- Gravitational Force work (U_g)

$\boxed{U_g = \overbrace{mg}^{w}\,\Delta h}$

$[J]$, $[kg]$, $[N/kg]$ 9.81, $y_2 - y_1$ $[m]$

- Spring Force work (U_s)

$\boxed{U_s = -\frac{1}{2}k(x_2^2 - x_1^2) = \frac{1}{2}k(x_1^2 - x_2^2)}$

k: spring constant

- Friction Force work (U_f)

$\boxed{U_f = -F_f \Delta x}$

F_f: kinetic fric force

- Kinetic Energy of a mass

$\boxed{T = \frac{1}{2}mv^2}$

$T_2 - T_1$

$\boxed{U_{1\to2} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2}$

- work / Energy Relationships

$\boxed{T_1 + U_{1\to2} = T_2}$

$\frac{1}{2}mv_1^2$; U_g, U_s, U_f, U_{app}; $\frac{1}{2}mv_2^2$

- Conservative Force - work done is PATH INDEPENDENT

1) $\boxed{U_g = -mg(y_2 - y_1) = -mgy_2 - (-mgy_1)}$

2) $\boxed{U_s = -\frac{1}{2}k(s_2^2 - s_1^2) = -\frac{1}{2}ks_2^2 - (-\frac{1}{2}ks_1^2)}$

} depend on initial + final position ONLY

vs

3) $\boxed{U_f = -F_f \, \Delta S_{total}}$ } depends on path (non-conservative)

3.7 Potential Energy

- Potential Energy (V) - Pot. Energy of mass at given location associated w/ Conservative force is work done on the mass by force in moving from current location to ref. point ('0').

- Gravitational Pot. Energy (V_g)

$\boxed{U_g = V_{g1} - V_{g2}}$

mgy_1, mgy_2

- Elastic Pot. Energy (V_e)

$\boxed{U_e = V_{e1} - V_{e2}}$

$\frac{1}{2}ks_1^2 - \frac{1}{2}ks_2^2$

- Potential work Energy Equation

1) w/ non-conservative force (ex: U_f)

$\boxed{T_1 + V_1 + U_{non\text{-}cons} = T_2 + V_2}$

$T_1 = \frac{1}{2}mv_1^2$; $V_1 = V_{g1} + V_{e1}$ (mgy_1, $\frac{1}{2}ks_1^2$); $T_2 = \frac{1}{2}mv_2^2$; $V_2 = V_{g2} + V_{e2}$ (mgy_2, $\frac{1}{2}ks_2^2$)

(Conservation of Energy)

2) w/ out non-conservative force (frictionless)

$\boxed{T_1 + V_1 = T_2 + V_2}$

$\frac{1}{2}mv_1^2$; $V_{g1} + V_{e1}$ (mgy_1, $\frac{1}{2}ks_1^2$); $\frac{1}{2}mv_2^2$; $V_{g2} + V_{e2}$ (mgy_2, $\frac{1}{2}ks_2^2$)

Note: Different methods:

Force & Acceleration
↳ $\vec{F} = m\vec{a}$ (acc, forces)

Work & Energy
↳ $\int F\cos\theta \, ds$, $\frac{1}{2}mv^2$ (vel, displacement)

Momentum & Impulse
↳ $m\vec{v}$, $\int F\,dt$ (vel, time)

3.9 Linear Impulse & Linear Momentum

- Linear momentum

$\boxed{\vec{L} = m\vec{v}}$

- Linear Impulse

$\boxed{L.I. = \int_{t_1}^{t_2} \Sigma \vec{F}\,dt}$

- Linear ~

$\boxed{m\vec{v}_1 + \underbrace{\int_{t_1}^{t_2} \Sigma \vec{F}\,dt}_{\boxed{F\Delta t}} = m\vec{v}_2}$

$\boxed{F\Delta t}$ } F = constant, $t_2 - t_1$

(conservation of Momentum)
- No external impulse ($\vec{F} = 0$)

$\Sigma m\vec{v}_1 = \Sigma m\vec{v}_2$

↳ $\Sigma(m_i v_i)_1 = \Sigma(m_i v_i)_2$

3.10 Angular Impulse & Angular Momentum

- Angular Momentum ($\vec{H}$)
  - w/ forces $M = \vec{r} \times \vec{F}$
  - w/ linear momentum: $\vec{H} = \vec{r} \times m\vec{v}$

$\boxed{\vec{H}_o = \vec{r} \times m\vec{v} = |\vec{r}||m\vec{v}|\sin\theta = mrv_\theta}$

angular momentum about point O $[kg \cdot \frac{m^2}{s}]$; distance betw P & O; polar coordinates.

- Principle of Angular Momentum & Impulse

$\boxed{(\vec{H}_o)_1 + \int_{t_1}^{t_2} \Sigma \vec{M}_o \, dt = (\vec{H}_o)_2}$

$r_1 m v_1$; $\boxed{M\Delta t}$ } M = constant; $r_2 m v_2$

(conservation of Angular Momentum)
- No external moment

$\Sigma_i [(\vec{H}_i)_o]_1 = \Sigma_i [(\vec{H}_i)_o]_2$

- Relation betw moment & angular momentum

$\boxed{\Sigma \vec{M}_o = \frac{d\vec{H}_o}{dt}}$

$\Sigma(r \times F)$; $\dot{H}_o$

- IMPORTANT EQUATIONS

$\boxed{\bar{r} = \frac{\Sigma m_i r_i}{\Sigma m_i}}$ → distance from O; m_i: masses

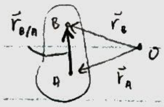
$\boxed{\dot{\bar{r}} = \frac{\Sigma m_i \dot{r}_i}{\Sigma m_i}} = \frac{(m_1 v_1)\hat{i} + (m_2 v_2)\hat{j} + (m_3 v_3)\hat{k}}{m_1 + m_2 + m_3}$

$\boxed{\ddot{\bar{r}} = \frac{\Sigma F}{\Sigma m_i}} = \frac{F\hat{i} + F\hat{j} + F\hat{k}}{m_1 + m_2 + m_3}$

$\boxed{H_o = \Sigma r_i \times m_i v_i} = (r_1 \times m_1 v_1) + (r_2 \times m_2 v_2) + (r_3 \times m_3 v_3) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ & & \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ & & \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ & & \end{vmatrix}$

$\boxed{\dot{H}_o = \Sigma M_o = \vec{r} \times \vec{F}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ & & \end{vmatrix}$

- Planar kinematics of rigid bodies.
  - Relies on relative motion



$\vec r_B = \vec r_A + \vec r_{B/A}$
$\vec v_B = \vec v_A + \vec v_{B/A}$
$\vec a_B = \vec a_A + \vec a_{B/A}$

- Types of motion
  - Translation : $\boxed{\vec v_B = \vec v_A}$ $\boxed{\vec a_B = \vec a_A}$

- Rotation : angular velocity $\boxed{\omega = \frac{d\theta}{dt}}$ ($\dot\theta$) → $\boxed{\alpha\, d\theta = \omega\, d\omega}$
  angular accele. $\boxed{\alpha = \frac{d\omega}{dt}}$ ($\ddot\theta$)

* ↺ $\vec\omega, \vec\alpha$ } $(+\hat k)$ ↻ $\vec\omega, \vec\alpha$ } $(-\hat k)$

* Constant Angular acceleration, α_c :
  $\boxed{\omega = \omega_0 + \alpha_c t}$
  $\boxed{\omega^2 = \omega_0^2 + 2\alpha_c \Delta\theta}$
  $\boxed{\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha_c t^2}$

Note: ω & α are properties of motion of entire body
↳ same for WHOLE body

- Have $\vec\omega$ & $\vec\alpha$ ⇝ want $\vec v_P$ & $\vec a_P$
- vel: $\boxed{\vec v_P = r_{P/O}\,\omega\,\hat u_\theta}$ } $\boxed{\vec v_P = \vec\omega \times \vec r_P}$
- acc: $\boxed{\vec a_P = \underbrace{-r\omega^2 \hat u_r}_{a_r} + \underbrace{r\alpha\,\hat u_\theta}_{a_\theta}}$ } $\boxed{\vec a_P = (\alpha \times r) + (\underbrace{\vec\omega \times \vec v_P}_{\vec\omega \times \vec r_P})}$
  or $\boxed{\vec a_P = \alpha \times r_{P/O} + \omega^2 \vec r_{P/O}}$

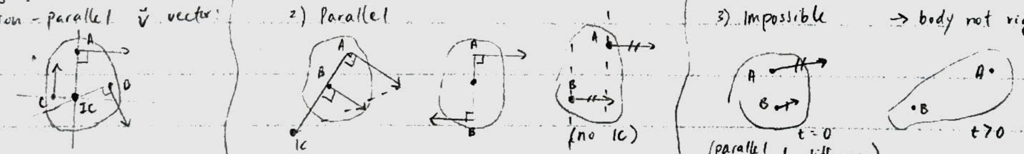
- Relative Motion Analysis
  - Velocity
    $\boxed{\vec v_B = \vec v_A + \vec v_{B/A}}$
    ↓ $\boxed{\vec v_B = \vec v_A + \vec\omega \times \vec r_{B/A}}$
    ↓ $\boxed{\vec v_{B/A} = \vec\omega \times \vec r_{B/A}}$
    ↳ direction ⊥ AB
    ↳ mag: $|\vec v_{B/A}| = \omega|\vec r_{B/A}|$

$|\omega| = \frac{|\vec v_{B/A}|}{|\vec r_{B/A}|}$ ↓ $\boxed{|\omega| = \frac{|\vec v_B - \vec v_A|}{|\vec r_{B/A}|}}$

- Acceleration
  $\boxed{\vec a_B = \vec a_A + \vec a_{B/A}}$ $+\,\vec\omega \times (\vec\omega \times \vec r_{B/A})$
  $\boxed{\vec a_B = \vec a_A + \underbrace{\vec\alpha \times \vec r_{B/A}}_{(\vec a_{B/A})_t} - \underbrace{\omega^2 \vec r_{B/A}}_{(\vec a_{B/A})_n}}$

$\boxed{|\alpha| = \frac{|\vec a_B - \vec a_A|}{|\vec r_{B/A}|}}$
↳ $\alpha = \frac{|(\vec a_{B/O})_t|}{|\vec r_{B/A}|} = \frac{|(\vec a_B)_t - (\vec a_A)_t|}{|\vec r_{B/A}|}$

- Instantaneous centre of 0 velocity (IC)
  ↳ lines ⊥ to velocity vectors of all points on a rigid body may intersect at a point ⇝ intersection = IC
- 2 cases :
  1) Non-parallel $\vec v$ vector: 2) Parallel 3) Impossible → body not rigid by will transform ✗



- Planar kinematics of rigid bodies
- Equations of motion
  ↶ $\boxed{H_G = I_G\,\omega}$
  $\boxed{M_G = \frac{dH_G}{dt} = I_G\,\alpha}$ ⇝ centre of mass G (well known G)
  $\boxed{M_O = I_O\,\alpha}$ ⇝ fixed axis at O (use if fixed axis)
  $\Sigma \vec F = m\vec a$ $\begin{cases} \Sigma F_x = m(a_G)_x \\ \Sigma F_y = m(a_G)_y \end{cases}$

* Deriving I : $I = J_P = \int_m \rho^2_{dm/P}\, dm$
  ↳ I (mass moment of inertia):
  ↳ physical constant describe object's ability to resist change in rotational speed

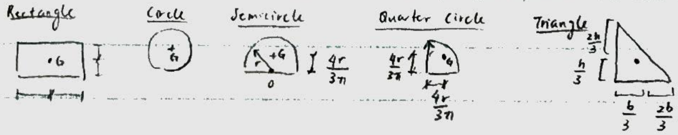
↱ Mass moment of inertia
I for common mass shapes :
1) Slender rod : $\boxed{I_A = \frac{1}{3}mL^2}$ $\boxed{I_G = \frac{1}{12}mL^2}$
2) Circle $\boxed{I_G = \frac{1}{2}mr^2}$
rolling wheel only: $\boxed{I_c = \frac{3}{2}mr^2}$

- radius of gyration: $\boxed{k_G = \sqrt{\frac{I_G}{m}}}$ or $\boxed{I_G = mk^2}$

- Finding center of mass (G)
  $\boxed{G = \frac{\Sigma(c\, m_i)}{\Sigma m_i}}$; c = coordinate of center of solid object
  ↳ $C_\triangle = (\frac{1}{3}b, \frac{1}{3}h)$ $C_\square = (\frac{1}{2}b, \frac{1}{2}h)$ $C_\circ = (r, r)$

- Parallel Axis Theorem
  ↳ to find I_P for point that is not centre of mass (G)
  $\boxed{I_P = J_G + md^2}$, $|PG|$
  ↳ point to center G
  $\boxed{I_G = \frac{1}{12}mL^2}$ $\boxed{I_G = \frac{1}{2}mr^2}$

d { Rectangle Circle Semicircle Quarter circle Triangle

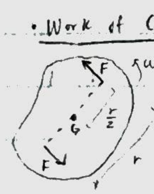


Work / Energy for Rigid Bodies
- Recal: $T_1 + V_1 + U_{1\to2} = T_2 + V_2$
- Work of a Force
  1) External applied force: $\boxed{U_F = \int \vec F \cdot d\vec r = \int_{s_1}^{s_2} F\cos\theta\, ds}$
  2) Weight $W = mg$: $\boxed{U_g = mg\,[(y_2)_G - (y_1)_G]}$
  3) Elastic Spring $\boxed{U_e = -\frac{1}{2}k(s_2^2 - s_1^2)}$

- Kinetic energy about fixed point 'O' $\boxed{T = \frac{1}{2}J_O\,\omega^2}$
  4) Kinetic Friction ↗ total path $\boxed{U_{Fr} = -\mu_k F_N (\Delta s)}$
  5) Static friction $\mu_s F_N$
  ↳ no work ($\Delta s = 0$)
  ↳ rolling wheel w/out slip
  ⇒ no work by F_{sf}

- Kinetic energy about moving centre of mass
  $\boxed{T = \underbrace{\frac{1}{2}I_G\,\omega^2}_{\text{rotation about G}} + \underbrace{\frac{1}{2}m v_G^2}_{\text{translation of G}}}$

- Work of Couple moment



$M = Fr$ ⇒ $\int dU_M = \int Fr\, d\theta = \int M\, d\theta$
$\boxed{U_M = M(\theta_2 - \theta_1)}$

Ex: $\ddot{x} + \frac{4k}{m}x = 0$

$w_n = \sqrt{\frac{4k}{m}}$

* Ex:
$\ddot{y} + 4w_n^2 y = 0$
$\hookrightarrow w_n = \sqrt{\frac{4k}{m}} = 2\sqrt{\frac{k}{m}}$

$C\sin(\underbrace{w_n t + \phi}_{rad})$

$C = \sqrt{A^2 + B^2}$

$\dot{x}_0 = B w_n$

$\phi = \tan^{-1}\left(\frac{A}{B}\right)$ rad

* $a_{max} = \mu_s g$ ($\ddot{x}$)

* $\boxed{w_n = \sqrt{\frac{k_{eq}}{m}}}$
* $\boxed{\delta_0 = \frac{F_0}{m}}$

• Conservation of Energy

Recall: $\boxed{T_1 + V_1 + (\Sigma U_{1\to 2})_{\text{non-conservative}} = T_2 + V_2}$

$\boxed{T_1 + V_1 = T_2 + V_2}$ — (no - non conservative forces)

• elastic potential E: $\boxed{V_e = \frac{1}{2}k s^2}$

• Grav. pot. E: $\boxed{V_g = mgh_G}$

• Principle of Moment & Impulse

$\boxed{m(\vec{v}_G)_1 + \int_{t_1}^{t_2} \Sigma \vec{F}\,dt = m(\vec{v}_G)_2}$ ← linear momentum

G: $\boxed{I_G w_1 + \int_{t_1}^{t_2} \Sigma M_G\,dt = I_G w_2}$ ← angular for rigid bodies

O: $\boxed{I_O w_1 + \int_{t_1}^{t_2} \Sigma M_O\,dt = I_O w_2}$

• Vibrations

• undamped → exclude friction
• free vibration → maintained by g or e
• forced vibration → external periodic force.

• Free undamped vibration

$\Sigma F_x = ma_x \Leftrightarrow F_s = -kx = m\ddot{x}$ (acceleration)

$\therefore \boxed{m\ddot{x} + kx = 0} \Leftrightarrow \boxed{\ddot{x} + \frac{k}{m}x = 0} \Rightarrow \boxed{\ddot{x} + w_n^2 x = 0}$

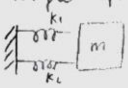
• natural frequency w_n $\left[\frac{rad}{s}\right]$ → $\boxed{w_n = \sqrt{\frac{k}{m}}}$ $= \left[\frac{1}{s}\right]$ (N/m, kg) $\boxed{= 2\pi f}$

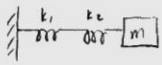
• frequency
$\boxed{f = \frac{w_n}{2\pi} = \frac{1}{T}}$ $\left[\frac{cycles}{s}\right]$

• Period (T)
$\boxed{T = \frac{2\pi}{w_n} = \frac{1}{f}}$ $\left[\frac{s}{cycles}\right]$

* $K = \frac{F}{\Delta x}$

• Multiple Springs

parallel  $\boxed{k_{eq} = k_1 + k_2}$

Series  $\boxed{\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}}$
$\hookrightarrow \boxed{k_{eq} = \frac{(k_1)(k_2)}{k_1 + k_2}}$

• Forced Vibrations. → δ_0

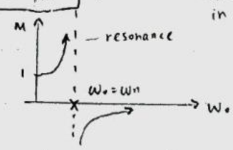
$\boxed{\ddot{x} + \underbrace{w_n^2}_{\left(\frac{k}{m}\right)x} x = \frac{F_0}{m}\sin(w_0 t)}$ periodic applied force function over m

$\boxed{x(t) = C\sin w_0 t}$

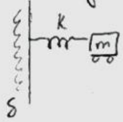
Amplitude: $\boxed{C = \frac{F_0}{k\left(1 - \left(\frac{w_0}{w_n}\right)^2\right)} = \frac{\delta_0}{1 - \frac{w_0^2}{w_n}}}$

* if start from equilib. ⇒ don't need to consider F_g b/c spring force accounts

* Magnification factor $\boxed{M = \frac{(x_p)_{max}}{F_0/k} = \frac{1}{1 - \left(\frac{w_0}{w_n}\right)^2}}$
↳ look at $\left(1 - \left(\frac{w_0}{w_n}\right)^2\right)^{-1}$
∘ if get $w_0 = w_n$, $M \to \infty$
↳ resonance & system fail →

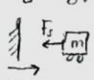


• Vibrating supports.



* look at moment when wall moves toward wall w/ displacement

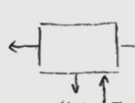
$\delta = \delta_0 \sin w_0 t$



$\vec{F}_s = -k(x - \delta_0 \sin w_0 t)$

$\boxed{\ddot{x} + \frac{k}{m}x = \frac{k}{m}\delta_0 \sin w_0 t}$
$\hookrightarrow x(t) = C\sin w_0 t$ $\boxed{C = \frac{\delta_0}{1 - \left(\frac{w_0}{w_n}\right)^2}}$ $\boxed{M = \frac{C}{\delta_0}}$

* FBD



$\Sigma F_x = ma_x$
$F_0 \sin w_0 t - kx = m\ddot{x}$

• Rigid Body Momentum and Impulse

• Recall: $\boxed{\vec{L} = m\vec{v}_G}$
scalar in 2D planar kinetic: $\boxed{H_G = I_G w}$, $\boxed{H_O = I_O w}$
$\boxed{\Sigma \vec{F} = m\vec{a}_G}$

i) Translation Only: $\boxed{\vec{L} = m\vec{v}_G}$, $\boxed{H_G = 0}$

ii) Rotation about fixed axis: $\boxed{\vec{L} = m\vec{v}_G}$, $\boxed{H_G = I_G w}$
OR
H about O: $\boxed{H_O = I_O w}$ $(I_O = I_G + md^2)$

iii) General plane motion: $\boxed{\vec{L} = m\vec{v}_G}$, $\boxed{H_G = I_G w}$, $\boxed{H_{IC} = I_{IC} w}$

• Conservation of Momentum

if $\boxed{\Sigma \int_{t_1}^{t_2} F\,dt = 0 \Rightarrow \vec{L}_1 = \vec{L}_2}$ $\{ (mv)_1 = (mv)_2$ ⇒ cons of linear momentum

if $\boxed{\Sigma \int_{t_1}^{t_2} M_G dt = 0 \Rightarrow (\vec{H}_G)_1 = (\vec{H}_G)_2}$ ⇒ cons. of angular momentum about G

if $\boxed{\Sigma \int_{t_1}^{t_2} M_O\,dt = 0 \Rightarrow (\vec{H}_O)_1 = (\vec{H}_O)_2}$ ⇒ conserve of angular momentum about fixed axis O

• General Solution

$\boxed{C = \sqrt{A + B}}$ $\boxed{\phi = \tan^{-1}\left(\frac{B}{A}\right)}$

$\boxed{x(t) = A\sin(w_n t) + B\cos(w_n t) = C\sin(w_n t + \phi)}$
A, B: constant of integration; C: amplitude (max angular displacement); ϕ: phase angle
→ complementary solution x_c for FREE vibration

$\boxed{x(t) = x_c + x_p}$ ↳ particular solution.

* for steady state solution $(t\to\infty)$, w_c = neglect.

• Undamped Vibration → Pendulum

$A = 0$ $B = \frac{v}{l w_n}$

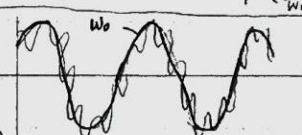
$\boxed{w_n = \sqrt{\frac{g}{l}}}$
$\hookrightarrow \boxed{T = \frac{2\pi}{w_n} = 2\pi\sqrt{\frac{l}{g}}}$

$\boxed{w_n = 1.22\sqrt{\frac{g}{l}}}$

* Full solution
* to find A → use $\dot{x}(0)$
* to find B → use $\ddot{x}(0)$

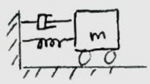
$\boxed{x(t) = x_c + x_p = \underbrace{A\sin(w_n t) + B\cos(w_n t)}_{x_c} + \underbrace{C\sin(w_0 t)}_{x_p}}$

$\hookrightarrow \boxed{x(t) = A\sin(w_n t) + B\cos(w_n t) + \frac{F_0/k}{1 - \left(\frac{w_0}{w_n}\right)^2}\sin(w_0 t)}$



⇒ Forced vibration, more interested in w_0 (affect more object)

· Damped Vibrations



Damping force

$\boxed{F_d = -c\dot{x}}$

↓ damping strength $\left[\frac{N \cdot s}{m}\right]$

· Differential Equation (by $\Sigma F = ma$)

$\boxed{m\ddot{x} + c\dot{x} + kx = 0}$

· General solution

$\boxed{x(t) = e^{\lambda t}}$

$$m\frac{d^2 e^{\lambda t}}{dt^2} + c\frac{d e^{\lambda t}}{dt} + ke^{\lambda t} = 0$$

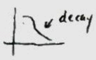
$$m\lambda^2 + c\lambda + k = 0 \rightsquigarrow \lambda = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

· 3 cases :

$(c = c_c)$

1) Critically damped : $\left(\frac{c}{2m}\right)^2 - \frac{k}{m} = 0$

$\boxed{C_{crit} = 2m\sqrt{\frac{k}{m}} = 2m\,w_n}$



↳ critical damping coefficient.

∘ Roots : $\boxed{\lambda_1 = \lambda_2 = \frac{-c}{2m} = -w_n}$

∘ Solution : 2 sol for $x(t)$

$\boxed{x(t) = (A + Bt)e^{-w_n t}}$ $\rightsquigarrow \lambda_1 = \lambda_2 = -w_n$

$\dot{x}(t) = (A + Bt)\lambda e^{\lambda t} + Be^{\lambda t}$

$\ddot{x}(t) = B\lambda e^{\lambda t} + (A + Bt)e^{\lambda}e^{\lambda t} + B\lambda e^{\lambda t}$

↳ combine into $m\ddot{x} + c\dot{x} + kx$ & match coeff.

$(c > c_c)$

2) Overdamped : $\left(\frac{c}{2m}\right)^2 - \frac{k}{m} > 0$

∘ Roots : λ_1 & λ_2 real values & -ve

$\boxed{\lambda_1 = \frac{-c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}}$

$\boxed{\lambda_2 = \frac{-c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}}$

∘ Solution :

$\boxed{x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}}$

$(c < c_c)$

3) Underdamped : $\left(\frac{c}{2m}\right)^2 - \frac{k}{m} < 0$

∘ Roots : λ_1, λ_2 imaginary

$\lambda_1, \lambda_2 = -\frac{c}{2m} \pm \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}\, i$ $\rightsquigarrow i = \sqrt{-1}$

∘ Solution :

$\boxed{x(t) = De^{-\frac{c}{2m}t}(\sin w_d t + \phi)}$

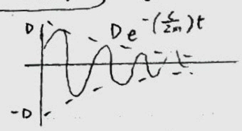
max amplitude

damping frequency [rad/s]

phase shift

$*\ \boxed{w_d = w_n\sqrt{1 - \left(\frac{c}{c_{crit}}\right)^2}} = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}$

$w_n = \sqrt{\frac{k}{m}}$



* solve D & ϕ by using $(x(0), \dot{x}(0))$ initial cond.

(ζ)

* $\frac{c}{c_c}$ = critical damping ratio $\rightsquigarrow$ use 2 x values @ diff time to find $\frac{c}{c_c}$

if $c \ll c_c \rightsquigarrow \boxed{\frac{x_1}{x_2} = \frac{Ce^{-\zeta w_n t_1}}{Ce^{-\zeta w_n (t_1 + T_d)}}} = e^{\zeta w_n T_d}$ → Period of damping ratio $\boxed{T_d = \frac{2\pi}{w_d}}$

↳ logarithmic decrement

$$\boxed{\delta = \ln\left(\frac{x_1}{x_2}\right) = \zeta w_n T_d = \zeta w_n \frac{2\pi}{w_n\sqrt{1 - \zeta^2}} = \frac{2\pi\zeta}{\sqrt{1 - \zeta^2}}}$$