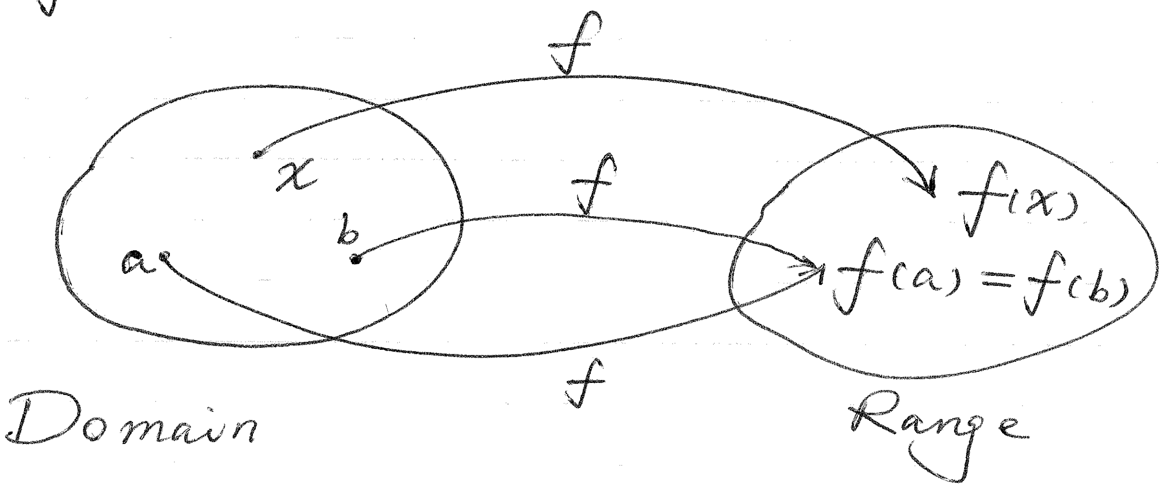
Chapter 1 Functions

1.1 Review of functions

① Relationships among quantities, or variables
② These relationships are expressed by mathematical objects called functions
③ Calculus is the study of functions

Def. A function f is a rule that assigns to each value x in a set D a unique value $f(x)$

The set D is the domain of the function. The range is the set of all values of $f(x)$ as x varies over the domain.



$y = f(x)$: x is the independent variable and y is the dependent variable.

The graph of a function f is the set of all points (x, y) in the xy plane that satisfy $y = f(x)$.

$f(\cdots\cdots)$: $\cdots\cdots$ is the argument of f

$f(x) = x^2 - 2x$ with different arguments :

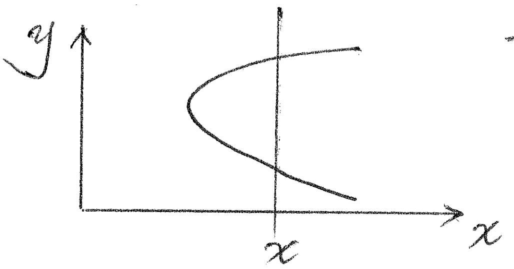
$f(-1) = (-1)^2 - 2(-1) = 3$

$f(t) = t^2 - 2t$

$f(p-1) = (p-1)^2 - 2(p-1) = p^2 - 4p + 3$

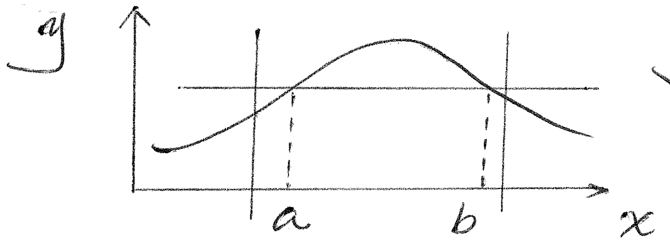
Does a graph represent a function ?

Vertical Line Test



To x , not a unique value of y is assigned.

Not a function .

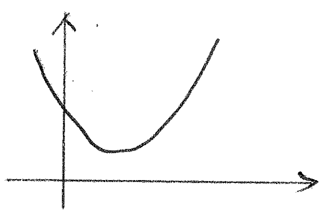


Yes, it represents a function .

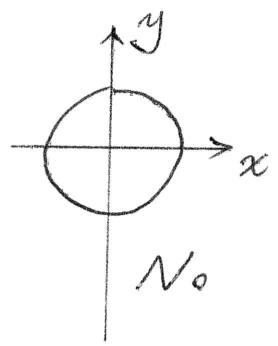
$f(a) = f(b)$ noted .

A unique value of y is assigned to each x .

Ex 1



Yes .



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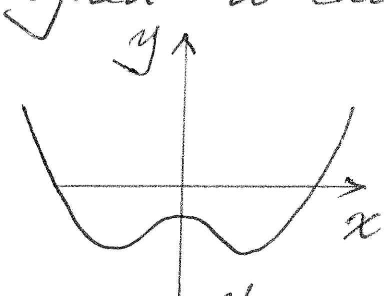


Yes

Ex 2 .

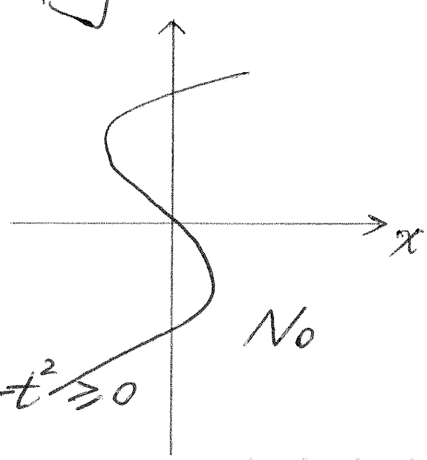
$y = f(x) = x^2 + 1$

Domain : $(-\infty, \infty)$, range : $[1, \infty)$

$z = g(t) = \sqrt{4 - t^2}$

Domain : $t \in [-2, 2]$ such that $4 - t^2 \geq 0$

Range : $z \in [0, 2]$



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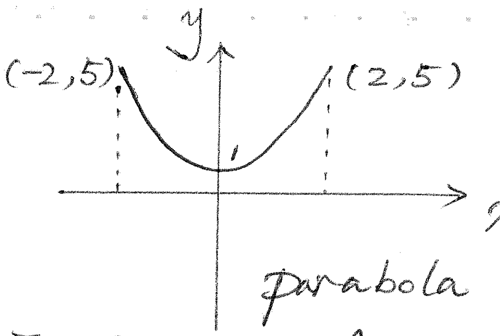
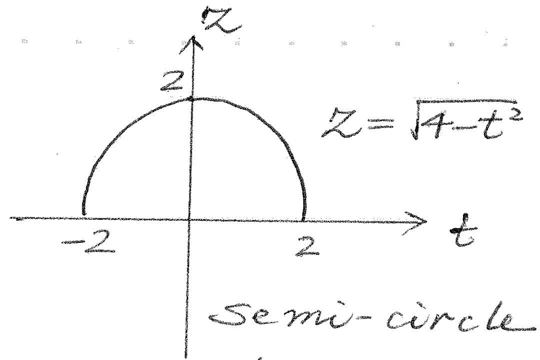


$y = x^2 + 1$



$z = \sqrt{4 - t^2}$

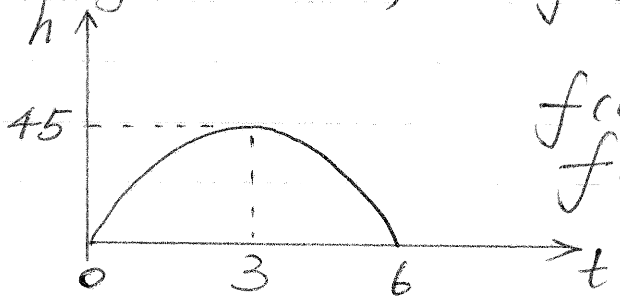
Ex 3 At $t=0$, a stone is thrown vertically upward from the ground at a speed of 30m/s. Its height above the ground (in meters) is approximated by

$$h = f(t) = 30t - 5t^2 \quad (t \text{ in sec})$$

Find the domain and range of this particular problem.

$30t - 5t^2 = 0$, $t = 0$ and 6

$f(0) = 0$, $f(6) = 0$



$f(0) = 0$, $f(3) = 45$ for max h, $f(6) = 0$

Domain : $t \in [0, 6]$

Range : $h \in [0, 45]$

Composite (復合) functions

Def: Given two functions f and g, the composite function $f \circ g$ is defined by $(f \circ g)(x) = f(g(x))$. It is evaluated in two steps : $y = f(u)$ with $u = g(x)$.

f is the outer function and g is the inner function.

The domain of $f \circ g$ consists of all x in the domain of g such that $u = g(x)$ is in the domain of f.

Ex 4

$f(x) = 3x^2 - x$ and $g(x) = \frac{1}{x}$

$f(5p+1) = 3(5p+1)^2 - (5p+1) = 75p^2 + 25p + 2$

$g(\frac{1}{x}) = \frac{1}{1/x} = x$

$f(g(x)) = 3(\frac{1}{x})^2 - \frac{1}{x} = \frac{3-x}{x^2}$

$g(f(x)) = \frac{1}{f(x)} = \frac{1}{3x^2 - x}$

Ex 5

$h(x) = \sqrt{9x - x^2}$

$h(x) = f(g(x))$

$f(u) = \sqrt{u}$, $u = g(x) = 9x - x^2$

domain: $9x - x^2 \geqslant 0$, $x(9-x) \geqslant 0$

$x \in [0, 9]$

$h(x) = \frac{2}{(x^2-1)^3}$

$h(x) = f(g(x))$

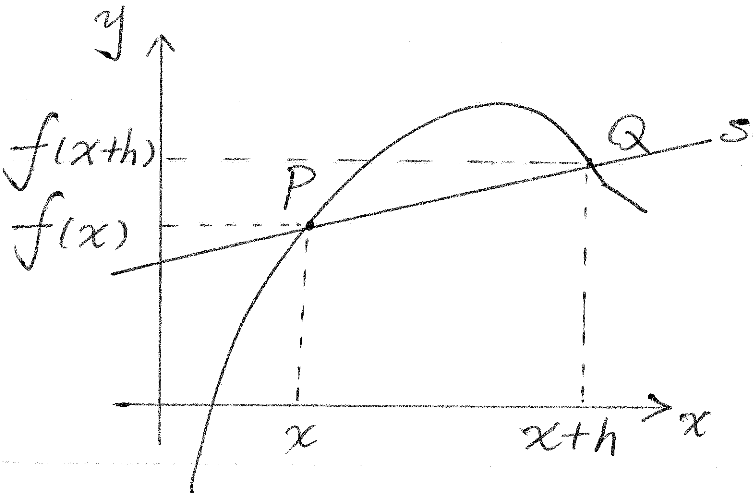
$f(u) = \frac{2}{u^3}$, $u = g(x) = x^2 - 1$

domain: $x^2 - 1 \neq 0$, $x \in \{x : x \neq \pm 1\}$.

all real values except 1 and -1.

Secant lines and difference quotients

Consider two points
$P\ (x, f(x))$ and $Q\ (x+h, f(x+h))$
on the graph of $y=f(x)$.
A line through any two points on a curve is called a <u>secant line</u> 割线



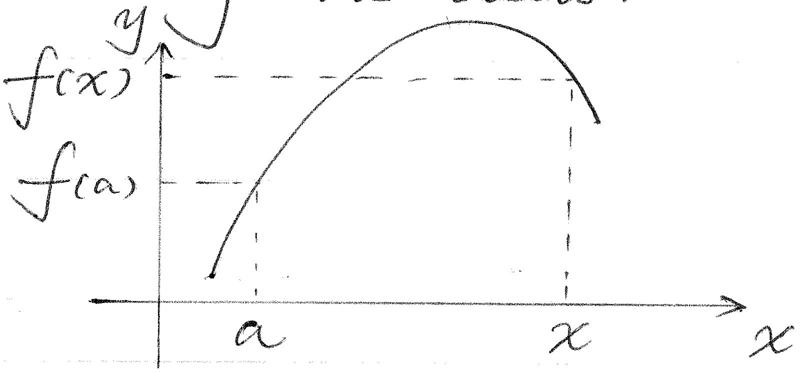
secant line

slope:

$$m_{sec} = \frac{\text{change in } y}{\text{change in } x} = \frac{f(x+h)-f(x)}{(x+h)-x} = \frac{f(x+h)-f(x)}{h}$$

m_{sec} is a difference quotient, describing average rate of change.

Change the labels.



$$m_{sec} = \frac{f(x)-f(a)}{x-a}$$

Ex 8. $\dfrac{f(x+h)-f(x)}{h}$ for $f(x)=3x^2-x$

$$m_{sec} = \frac{3(x+h)^2-(x+h)-(3x^2-x)}{h}$$

$$= \frac{\cancel{3x^2}+6xh+3h^2\cancel{-x}-h\cancel{-3x^2}\cancel{+x}}{h} = 6x+3h-1$$

$$\frac{f(x)-f(a)}{x-a} \quad \text{for} \quad f(x)=x^3$$

$$m_{sec} = \frac{x^3-a^3}{x-a} = \frac{(x-a)(x^2+ax+a^2)}{x-a} = x^2+ax+a^2$$

Ex 9 A sound source with acoustic power P (in Watts).
Sound intensity $I(r)$ at a point r meters away from the source is given by $I(r) = \dfrac{P}{4\pi r^2}$ (in W/m^2)

a. $r_1 = 10\ m$, $r_2 = 15\ m$, $P = 100\ W$

$$I(r_1) = \frac{100}{4\pi(10)^2} = \frac{1}{4\pi}\ W/m^2$$

$$I(r_2) = \frac{100}{4\pi(15)^2} = \frac{1}{9\pi}\ W/m^2$$

$$m_{sec} = \frac{\frac{1}{9\pi} - \frac{1}{4\pi}}{15-10} = \frac{\frac{-5}{36\pi}}{5} = -\frac{1}{36\pi}\ (W/m^2)/m$$

b. general expression

$$m_{sec} = \frac{I(r_2)-I(r_1)}{r_2-r_1} = \frac{\frac{P}{4\pi r_2^2} - \frac{P}{4\pi r_1^2}}{r_2-r_1}$$

$$= \frac{P}{4\pi} \frac{1}{r_1^2 r_2^2} \frac{r_1^2 - r_2^2}{r_2-r_1} = -\frac{P}{4\pi} \frac{r_1+r_2}{r_1^2 r_2^2}$$

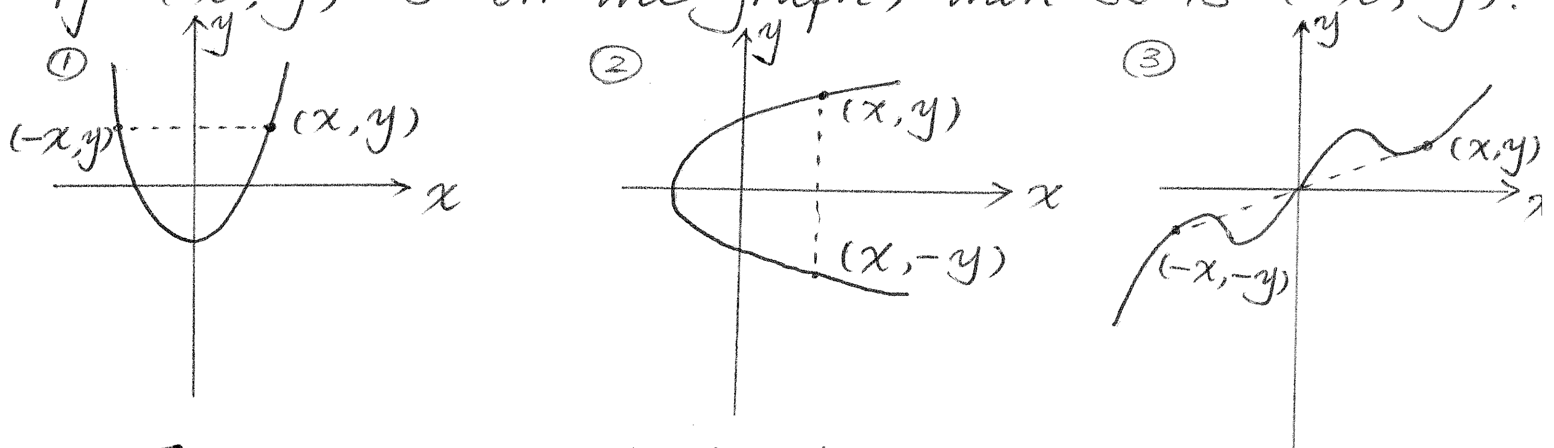
$m_{sec} < 0$: $I(r)$ decreases with the increasing r.

Symmetry of graphs

1. A graph is symmetric w.r.t. the y axis $\leftrightarrow$ If (x,y) is on the graph, then so is $(-x,y)$.

2. A graph is symmetric w.r.t. the x axis $\leftrightarrow$ If (x,y) is on the graph, then so is $(x,-y)$.

3. A graph is symmetric w.r.t. the origin $\leftrightarrow$ If (x,y) is on the graph, then so is $(-x,-y)$.



Symmetry about both the x and y axes $\overset{\Longrightarrow}{\not\Longleftarrow}$ Symmetry about the origin

Even functions $\quad f(-x)=f(x)$

$y=x^2$

If (x,y) is on the graph with $y=f(x)$, then $(-x,y)$ is also on the graph. $\rightarrow$ Symm. about the y axis.

Odd functions $\quad f(-x)=-f(x)$

$y=x^3$

If (x,y) is on the graph with $y=f(x)$, then $(-x,-y)$ is also on the graph $\rightarrow$ Symm. about the origin.