MAT 2384 A - Fall 2020 - MIDTERM #1

Family Name ________________

Given Name solutions

Student Number ________________

- This is a closed book exam. You can have one sheet (one side) as a memory aid during the midterm.
- Only basic scientific calculators are allowed. Graphing or programmable calculators are not permitted.
- The exam has 4 questions worth a total of 22 points.
- The exam has 9 pages.
- You must answer all the questions.
- You have 90 minutes to submit the exam in Brightspace (see detailed instructions in Brightspace).
- Please scan and submit your solutions in Brightspace using one file in pdf format. No photos and no zip files can be accepted.

Question 1. [6 points] Solve the following IVP:

$$(3y\cos x + 2xy^2)dx + (4y + 9\sin x + 4x^2y)dy = 0, \quad y(0) = 1.$$

$M(x,y) = 3y\cos x + 2xy^2 \Rightarrow M_y = 3\cos x + 4xy$

$N(x,y) = 4y + 9\sin x + 4x^2y \Rightarrow N_x = 9\cos x + 8xy$ } $M_y \neq N_x$ DE not exact

$\frac{M_y - N_x}{M} = \frac{-6\cos x - 4xy}{3y\cos x + 2xy^2} = \frac{-2(3\cos x + 2xy)}{y(3\cos x + 2xy)} = \frac{-2}{y}$ function of y only

$\mu(y) = e^{-\int -\frac{2}{y}dy} = e^{2\ln y} = y^2$ and the DE becomes

$(3y^3\cos x + 2xy^4)dx + (4y^3 + 9y^2\sin x + 4x^2y^3)dy = 0$

$M^*(x,y) = 3y^3\cos x + 2xy^4 \Rightarrow M^*_y = 9y^2\cos x + 8xy^3$

$N^*(x,y) = 4y^3 + 9y^2\sin x + 4x^2y^3 \Rightarrow N^*_x = 9y^2\cos x + 8xy^3$ } $M_y^* = N_x^*$ DE now exact

$F(x,y) = \int M^*(x,y)dx + g(y)$ (or $\int N^*(x,y)dy + g(x)$)

$= \int(3y^3\cos x + 2xy^4)dx + g(y) = 3y^3\sin x + x^2y^4 + g(y)$

then $\frac{\partial F}{\partial y} = 9y^2\sin x + 4x^2y^3 + g'(y) = N^*(x,y) = 4y^3 + 9y^2\sin x + 4x^2y^3$

thus $g'(y) = 4y^3 \Rightarrow g(y) = y^4$

and the general solution is $3y^3\sin x + x^2y^4 + y^4 = C$

then $y(0) = 1 \Rightarrow 3(1)^3\sin(0) + (0)^2(1)^4 + (1)^4 = C \Rightarrow C = 1$

$\therefore$ the unique solution is $\boxed{3y^3\sin x + x^2y^4 + y^4 = 1}$

Question 2. [5 points] Solve the following IVP:

$$y' + y\tan x = y^2, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}, \quad y(0) = \frac{1}{2}$$

this is a Bernoulli equation with $p(x) = \tan x$, $g(x) = 1$, $a = 2$

so we let $u = y^{1-a} = y^{-1}$ and the DE will become

$$u' + (-1)u\tan x = -1$$

or $u' - u\tan x = -1$ which is linear with $p(x) = -\tan x$, $r(x) = -1$

then $\mu(x) = e^{\int -\tan x \, dx} = e^{\ln \cos x} = \cos x$

and then $u(x) = \frac{1}{\cos x}\left[\int (-1)\cos x \, dx + C\right]$

$$= \frac{1}{\cos x}(-\sin x + C)$$

so $y(x) = \frac{1}{u(x)} = \frac{\cos x}{C - \sin x}$ (general solution)

but $y(0) = \frac{1}{2} \Rightarrow \frac{1}{2} = \frac{\cos(0)}{C - \sin(0)} = \frac{1}{C} \Rightarrow C = 2$

$\therefore$ unique solution is $\boxed{y(x) = \frac{\cos x}{2 - \sin x}}$

Question 3. [6 points] Find the general solution of each of the following ODEs:

(a) $x^2y'' - xy' + y = 0, \quad x > 0.$

(b) $y'' - 6y' + 9y = 0.$

(c) $y'' - 8y' + 17y = 0.$

a) the char. eq. is $m(m-1) - m + 1 = m^2 - 2m + 1 = 0$

or $(m-1)^2 = 0 \Rightarrow m_1 = m_2 = 1$

the general solution is $\boxed{y(x) = C_1 x + C_2 x \ln x}$

b) the char. eq. is $\lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = 3$

the general solution is $\boxed{y(x) = C_1 e^{3x} + C_2 x e^{3x}}$

c) the char. eq. is $\lambda^2 - 8\lambda + 17 = 0$

$$\lambda_{1,2} = \frac{8 \pm \sqrt{(-8)^2 - 4(17)}}{2} = \frac{8 \pm \sqrt{-4}}{2} = 4 \pm i$$

the general solution is

$\boxed{y(x) = C_1 e^{4x} \cos x + C_2 e^{4x} \sin x}$

Question 4. [5 points] Consider the function $f(x) = x^4 + 7x - 6$.

(a) ([**1 point**]) Use the intermediate Value Theorem to prove that $f(x)$ has a root in the interval $[0, 1]$.

(b) ([**2 points**]) Rewrite the equation $f(x) = 0$ under the form $g(x) = x$ for some function $g(x)$ that satisfies the conditions for the convergence of the iteration sequence $x_{n+1} = g(x_n)$ (Verify that the conditions are satisfied).

(c) ([**2 points**]) Use the **fixed point iteration** method to find the root of the function $f(x)$ in the interval $[0, 1]$ to 4 decimal places, starting with $x_0 = 0.75$.

a) $\left.\begin{matrix} f(0) = -6 \\ f(1) = 2 \end{matrix}\right\}$ so $f(x)$ has changed sign over interval (and $f(x)$ is continuous) $\Rightarrow$ root in $[0,1]$

b) if $x^4 + 7x - 6 = 0 \Rightarrow x = \frac{6 - x^4}{7} = g(x)$

then $|g'(x)| = \left| \frac{-4x^3}{7} \right| = \frac{4}{7} x^3 \le \frac{4}{7} < 1$ on $[0,1]$

($g(x)$ is increasing $\Rightarrow$ max @ $x = 1$)

so convergence condition is met

c) $x_0 = 0.75$, $x_1 = \frac{6 - (0.75)^4}{7} = 0.8119$

$x_2 = 0.7951$ $\quad$ $x_5 = 0.7990$

$x_3 = 0.8000$ $\quad$ $x_6 = 0.7989 = x_7$ (stop)

$x_4 = 0.7986$

$\therefore$ root is $\boxed{0.7989}$

(check: $f(0.7989) \simeq -3.5 \times 10^{-4}$)