8.1

a) $-\int f(x) \ln f(x)\, dx$

$$= -\int f(x) \ln [\lambda e^{-\lambda x}]\, dx$$

$$= -\int f(x) [\ln \lambda - \lambda x]\, dx$$

$$= -\ln \lambda + \lambda E[X]$$

$$= -\ln \lambda + \lambda \lambda^{-1}$$

$$= 1 - \ln \lambda \qquad \text{nats}$$

b) $-\int f(x) \ln f(x)\, dx$

$$= -\int f(x) [\ln \tfrac{1}{2}\lambda - \lambda |x|]\, dx$$

$$= -\ln \frac{\lambda}{2} + \lambda E|X|$$

$$E|X| = 2\int_0^\infty \frac{1}{2} \lambda e^{-\lambda x} \cdot x\, dx$$

$$= \lambda^{-1}$$

$$\Rightarrow \quad h(X) = 1 + \ln \frac{2}{\lambda} \qquad \text{nats}$$

c) Let $X = X_1 + X_2$

Then $X \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$

$\Rightarrow h(X) = \frac{1}{2}\log[2\pi e(\sigma_1^2+\sigma_2^2)]$ bits.

8.2 $\underline{Z} = \underline{X}_\theta$

where $\underline{X}_1 \sim N(0, K_1)$

$\underline{X}_2 \sim N(0, K_2)$

Now, $h(\underline{Z}|\theta) = \lambda h(\underline{Z}|\theta=1) + (1-\lambda) h(\underline{Z}|\theta=2)$

$= \lambda h(\underline{X}_1) + (1-\lambda) h(\underline{X}_2)$

$= \lambda \frac{1}{2} \log (2\pi e)^n |K_1| + (1-\lambda) \frac{1}{2} \log (2\pi e)^n |K_2|$

Also, $E[\underline{Z}^H \underline{Z}] = \lambda E[\underline{Z}\underline{Z}^H|\theta=1] + (1-\lambda) E[\underline{Z}\underline{Z}^H|\theta=2]$

$= \lambda \cdot E[\underline{X}_1 \underline{X}_1^H] + 1-\lambda\, E(\underline{X}_2 \underline{X}_2^H]$

$= \lambda K_1 + (1-\lambda) K_2$

and $h(\underline{Z}) \leq \frac{1}{2} \log (2\pi e)^n |\lambda K_1 + (1-\lambda) K_2|$

$\Rightarrow$ Since $h(\underline{Z}|\theta) \leq h(\underline{Z})$, then

$\lambda \frac{1}{2} \log (2\pi e)^n + \frac{(1-\lambda)}{2} \log (2\pi e)^n + \frac{1}{2} \log |K_1|^\lambda + \frac{1}{2} \log |K_2|^{1-\lambda}$

$\leq \frac{1}{2} \log (2\pi e)^n + \frac{1}{2} \log |\lambda K_1 + (1-\lambda) K_2|$

Cancelling the $\frac{1}{2}\log(2\pi e)^n$ on both sides:

$$\cancel{\tfrac{1}{2}}\log|K_1|^{\lambda} + \cancel{\tfrac{1}{2}}\log|K_2|^{1-\lambda} \leq \cancel{\tfrac{1}{2}}\log|\lambda K_1 + (1-\lambda)K_2|$$

$$\Rightarrow \quad |K_1|^{\lambda}\,|K_2|^{1-\lambda} \leq |\lambda K_1 + (1-\lambda)K_2|$$

8.3

a) $I(X;Y) = h(Y) - h(Y|X)$

$$h(Y|X) = \int_{-\infty}^{\infty} h(Y|X=x) f_X(x)\,dx$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} h(Y|X=x) \cdot 1\,dx$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} h(Z+x)\,dx$$

$$= \int_{-1/2}^{1/2} \log a\,dx$$

$$= \log a$$

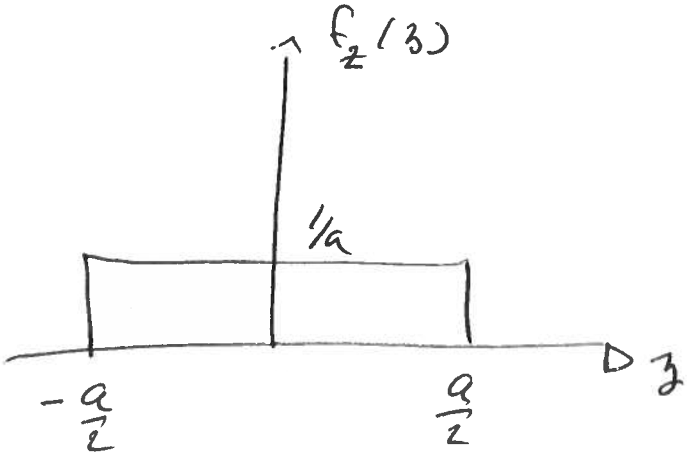
Since $h(Z+x) = h(Z) = \log a$.

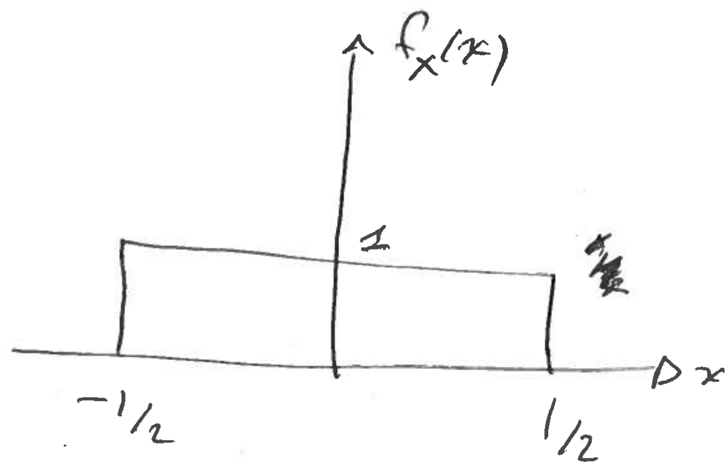
To find $h(Y)$ we need the pdf $f_Y(y)$ of Y.

Now $Y = X + Z$

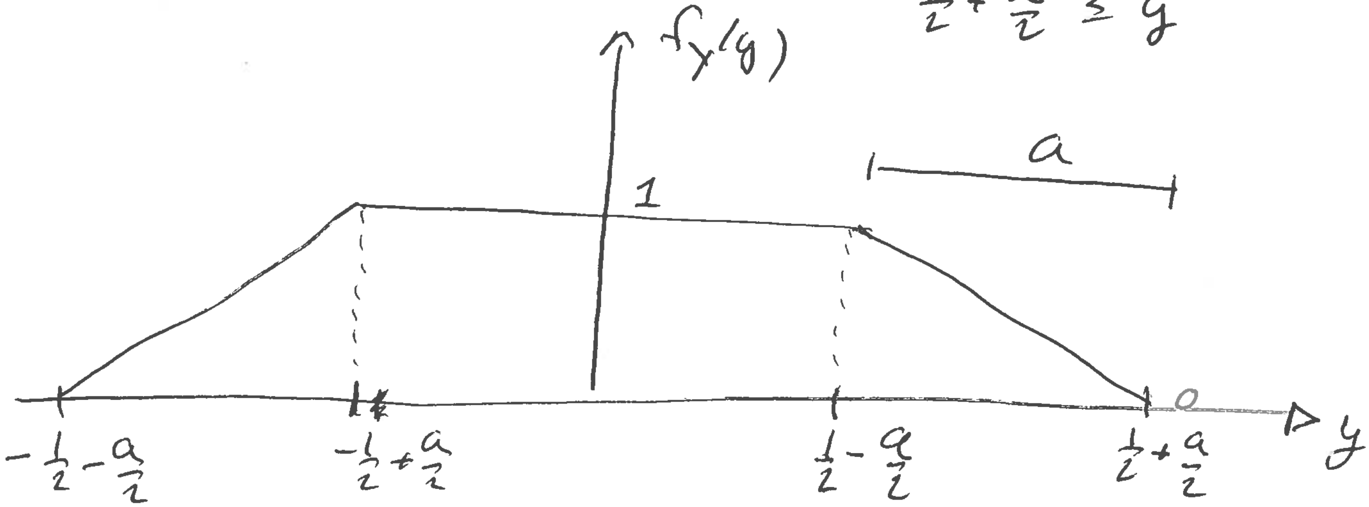
$$\Rightarrow f_Y(y) = \int_{-\infty}^{\infty} f_X(y-u)\, f_Z(u)\, du$$

The convolution is best done graphically:





Then
$$f_Y(y) = \begin{cases} 0 & y \le -\frac{1}{2} - \frac{a}{2} \\ \frac{1}{a}\left(y + \frac{1}{2} + \frac{a}{2}\right) & -\frac{1}{2} - \frac{a}{2} \le y \le -\frac{1}{2} + \frac{a}{2} \\ 1 & -\frac{1}{2} + \frac{a}{2} \le y \le \frac{1}{2} - \frac{a}{2} \\ \frac{1}{a}\left(\frac{1}{2} + \frac{a}{2} - y\right) & \frac{1}{2} - \frac{a}{2} \le y \le \frac{1}{2} + \frac{a}{2} \\ 0 & \frac{1}{2} + \frac{a}{2} \le y \end{cases}$$



$$h(Y) = -\int_{-\infty}^{\infty} f_Y(y) \log f_Y(y)\, dy$$

$$= -2\int_0^{\frac{1}{2}-\frac{a}{2}} f_Y(y) \log f_Y(y)\, dy - 2\int_{\frac{1}{2}-\frac{a}{2}}^{\frac{1}{2}+\frac{a}{2}} f_Y(y) \log f_Y(y)\, dy$$

$$\int_0^{\frac{1}{2}-\frac{a}{2}} f_Y(y) \log f_Y(y)\, dy = 0 \quad \text{since} \quad \log f_Y(y) = \log 1 = 0$$

$$\int_{\frac{1}{2}-\frac{a}{2}}^{\frac{1}{2}+\frac{a}{2}} f_Y(y) \log f_Y(y)\, dy = \int_0^a \left(\frac{u}{a}\right) \log\left(\frac{u}{a}\right) du \quad \Big| \quad v = \frac{u}{a}, \quad dv = \frac{du}{a}$$

$$= \int_0^1 (v \log v)(a\, dv)$$

$$= a \cdot \left[\frac{v^2 \log v}{2} - \frac{v^2}{4}\right]_0^1 = \frac{-a}{4}$$ if natural log is used.

$$\Rightarrow h(Y) = -2 \times \frac{-a}{4} = \frac{a}{2}$$

$$I(X;Y) = h(Y) - h(Y|X)$$
$$= \frac{a}{2} - \ln a$$