4.2 $$H(X_0 \mid X_{-n}^{-1}) = H(X_0 \mid X_{-n}^{-1}) + H(X_{-n}^{-1}) - H(X_{-n}^{-1})$$

$$= H(X_{-n}^{0}) - H(X_{-n}^{-1})$$

$$= H(X_0^{n}) - H(X_1^{n})$$

$$= H(X_0 \mid X_1^{n})$$

4.3) $$H(TX) \geq H(TX \mid T) \quad (a)$$

$$= H(T^{-1}TX \mid T) \quad (b)$$

$$= H(X \mid T) \quad (c)$$

$$= H(X) \quad (d)$$

(a) Conditioning reduces entropy

(b) given T, $H(Y \mid T) = H(T^{-1}Y \mid T)$ since the operation of multiplying by T^{-1} can be ~~inverse~~ inversed itself.

(c) Since T^{-1} is the inverse shuffle of T, then $T^{-1}TX = X$.

(d) Since ~~[illegible]~~ X & T are independent.

4.6 a) $(n-1)H(X_1^n) - nH(X_1^{n-1})$

$$= (n-1)H(X_1^{n-1}) + (n-1)H(X_n|X_1^{n-1}) - (n-1)H(X_1^{n-1}) - H(X_1^{n-1})$$

$$= [H(X_n|X_1^{n-1}) - H(X_1)]$$
$$+ [H(X_n|X_1^{n-1}) - H(X_2|X_1)]$$
$$+ \cdots$$
$$+ [H(X_n|X_1^{n-1}) - H(X_{n-1}|X_1^{n-2})]$$
$$\leq 0$$

Since each term in each square bracket is ≤ 0 as the process is stationary.

This is because $H(X_\ell | X_1^{\ell-1})$

$$= H(X_n | X_{n-\ell+1}^{n-1}) \geq H(X_n | X_1^{n-1})$$

for $\ell \leq n$.

4.6 b) $H(X_1^n) - nH(X_n | X_1^{n-1})$

$$= [H(X_1) - H(X_n | X_1^{n-1})]$$
$$+ [H(X_2 | X_1) - H(X_n | X_1^{n-1})]$$
$$+ \cdots$$
$$+ [H(X_n | X_1^{n-1}) - H(X_n | X_1^{n-1})]$$
$$\geq 0$$

since each term in the square brackets is ≥ 0.

4.9) We have by assumption that

$$X_0 \to X_{n-1} \to X_n$$

$$\Rightarrow \quad I(X_0; X_{n-1}) \geq I(X_0; X_n)$$

$$H(X_0) - H(X_0 | X_{n-1}) \geq H(X_0) - H(X_0 | X_n)$$

$$\Rightarrow \quad H(X_0 | X_n) \geq H(X_0 | X_{n-1})$$

4.10 a) X_i & X_j are independent by assumption when $1 \leq i < j \leq n-1$.

This leaves the case that $j = n$, i.e., the case of X_i & X_n with $1 \leq i \leq n-1$.

Now $X_n = X_1 \oplus X_2 \oplus \cdots \oplus X_{n-1}$
where $\oplus$ is mod 2 addition

Let $Z_i = X_n \oplus X_i$

$$= X_1 \oplus \cdots \oplus X_{i-1} \oplus X_{i+1} \oplus \cdots \oplus X_{n-1}$$

Then $P[Z_i = z \mid X_i = x] = 1/2$ for all $x, z \in \{0,1\}$
since Z_i does not depend on X_i.

$\Rightarrow P[X_n = x_n \mid X_i = x_i]$

$= P[Z_i = x_n \oplus x_i \mid X_i = x_i]$

$= P[Z_i = x_n \oplus x_i] = 1/2$ for all x_n & $x_i \in \{0,1\}$

b) Since $P[X_i = x_i, X_j = x_j] = \frac{1}{4}$

for $i \neq j$

then $H(X_i, X_j) = 2$

c) $H(X_1^n) = H(X_1^{n-1}) + H(X_n | X_1^{n-1})$

$= n-1 \quad + 0$

$= n-1$

$\neq n = nH(X_1)$

4.11 a) $H(X_n | X_0) = H(X_n | X_0) + H(X_0) - H(X_0)$

$$= H(X_n, X_0) - H(X_0)$$

$$= H(X_0, X_{-n}) - H(X_0)$$

$$= H(X_{-n} | X_0)$$

b) False. Let X_0 & X_1 be iid ~~uniform~~ uniform Bernoulli r.v.'s.

Let $X_l = \begin{cases} X_0 & \text{if } l \text{ even} \\ X_1 & \text{if } l \text{ odd} \end{cases}$

Then $\{X_i\}$ is stationary.

But $0 = H(X_2 | X_0) \leq H(X_1 | X_0) = H(X_1) = 1$

c) $H(X_{n+1} | X_1^n, X_{n+2}) = H(X_n | X_0^{n-1}, X_{n+1})$

$\leq H(X_n | X_1^{n-1}, X_{n+1})$

d) $H(X_{(n+1)} | X_1^n, X_{(n+1)+1}^{2(n+1)})$

$= H(X_n | X_0^{n-1}, X_{n+1}^{2n+1})$

$\leq H(X_n | X_1^{n-1}, X_{n+1}^{2n})$

4.12 For $\ell \geq 2$, $H(X_\ell | X_0, X_1, \ldots, X_{\ell-1})$

a)

$$= H(X_\ell | X_{\ell-1}\, X_{\ell-2})$$

Since the direction can be determined from $X_{\ell-1}$ & $X_{\ell-2}$.

Therefore

$$H(X_1, \ldots, X_n) = H(X_0, X_1, \ldots X_n) \quad \text{since } X_0 = \text{constant}$$

$$= H(X_0) + H(X_1 | X_0) + H(X_2 | X_1\, X_0)$$
$$+ H(X_3 | X_2\, X_1)$$
$$+ \cdots$$
$$+ H(X_n | X_{n-1}\, X_{n-2})$$

$$= 0 + h(1/2) + (n-1)\, h(p)$$

$$= 1 + (n-1)\, h(p)$$

b) $$H(\mathcal{X}) = \lim_{n \to \infty} \frac{1}{n} H(X_1^n) = \lim_{n \to \infty} \frac{1}{n} + \frac{n-1}{n} h(p)$$

$$= h(p).$$

4.13 $I(X_1^n; X_{n+1}^{2n})$

$$= H(X_{n+1}^{2n}) - H(X_{n+1}^{2n} | X_1^n)$$

$$= H(X_1^n) - H(X_{n+1}^{2n} | X_1^n)$$

Now $\lim_{n\to\infty} \frac{1}{n} H(X_1^n) = H(\mathbb{X})$

and

$$\lim_{n\to\infty} \frac{1}{n} H(X_{n+1}^{2n} | X_1^n)$$

$$= \lim_{n\to\infty} \frac{1}{n} \Big[H(X_{n+1} | X_1^n) + H(X_{n+2} | X_1^{n+1}) + \cdots + H(X_{2n} | X_1^{2n-1}) \Big]$$

$$= H'(\mathbb{X})$$

since $\lim_{n\to\infty} H(X_n | X_1^{n-1}) = H'(\mathbb{X})$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{2n} I(X_1^n; X_{n+1}^{2n}) = \frac{1}{2}\big[H(\mathbb{X}) - H'(\mathbb{X})\big]$$

$$= 0$$

since $H(\mathbb{X}) = H'(\mathbb{X})$ for a stationary process.

4.15

$$\frac{1}{n} H(X_n, \ldots, X_1 \mid X_{-k}^{0})$$

$$= \frac{1}{n} \Big[H(X_1 \mid X_{-k}^{0}) + H(X_2 \mid X_{-k}^{1}) + \cdots + H(X_n \mid X_{-k}^{n-1}) \Big]$$

$$= \frac{1}{n} \Big[H(X_{k+1} \mid X_0^{k}) + H(X_{k+2} \mid X_0^{k+1}) + \cdots + H(X_{n+k} \mid X_0^{n+k-1}) \Big]$$

$$\to H'(\mathbb{X})$$

Since $H'(X) = \lim_{l \to \infty} H(X_l \mid X_0^{l-1})$

Since $H'(\mathbb{X}) = H(\mathbb{X})$, the proof is complete.

4.23

a) Let $b_n = \frac{1}{n} H(X_1^n)$

Now $b_n = \frac{1}{n}[H(X_1) + H(X_2|X_1) + \cdots + H(X_n|X_1^{n-1})]$

$= \frac{1}{n}[H(X_0) + H(X_0|X_{-1}) + \cdots + H(X_0|X_{-n+1}^{-1})]$

$= \frac{1}{n}[a_0 + a_1 + \cdots + a_{n-1}]$

Where $a_\ell = H(X_0 | X_{-\ell}^{-1})$

and a_ℓ is thus decreasing in ℓ.

Therefor $b_n = \frac{1}{n}[a_0 + \cdots + a_{n-1}]$ is decreasing in n,

and $b_1 = \frac{1}{1} a_0 = H(X_0) = H(X_1)$

$\Rightarrow \lim_{n \to \infty} b_n \leq b_1 = H(X_1)$

b) We have equality iff $b_1 = b_2 = \cdots = b_n = \cdots$

$\Rightarrow H(X_0 | X_{-\ell}^{-1}) = H(X_0)$ or equivalently

$H(X_\ell | X_0^{\ell-1}) = H(X_0)$ for all ℓ

which is true if X_ℓ is independent of $X_0^{\ell-1}$ for all ℓ

which is true iff $\{X_i\}$ is a sequence of independent random variables.

4.24

a) $$H(\mathcal{Y}) = \lim_{n\to\infty} \frac{1}{n} H(Y_1, \ldots, Y_n)$$

$$= \lim_{n\to\infty} \frac{1}{n} H(X_1, \ldots, X_{n+1})$$

$$= \lim_{n\to\infty} \frac{1}{n} H(X_1^n) + \frac{1}{n} H(X_{n+1} \mid X_1^n)$$

$$= H(\mathcal{X})$$

b) $$H(\mathcal{Z}) = \lim_{n\to\infty} \frac{1}{n} H(Z_1, \ldots, Z_n)$$

$$= \lim_{n\to\infty} \frac{1}{n} H(X_2, X_3, \ldots, X_{2n+1})$$

$$= \lim_{n\to\infty} \frac{1}{n} H(X_1, \ldots, X_{2n})$$

$$= 2 \lim_{n\to\infty} \frac{1}{2n} H(X_1^{2n})$$

$$= 2 H(\mathcal{X})$$

c) $$H(\mathcal{V}) = \lim_{n\to\infty} \frac{1}{n} H(V_1^n)$$

$$= \lim_{n\to\infty} \frac{1}{n} H(X_2, X_4, \ldots, X_{2n})$$

$$= \lim_{n\to\infty} \frac{1}{n} \Big[H(X_2) + H(X_4 | X_2) + \cdots + H(X_{2n} | X_{2n-2}, X_{2n-4}, \ldots X_6, X_4, X_2) \Big]$$

$$\geq \lim_{n\to\infty} \frac{1}{n} \Big[H(X_2 | X_1) + H(X_4 | X_1^3) + \cdots + H(X_{2n} | X_1^{2n-1}) \Big]$$

$$= H'(X) = H(\mathcal{X})$$

d) Same as (b)