3.2

$$\frac{1}{n}\log\frac{p(\underline{X})\,p(\underline{Y})}{p(\underline{X},\underline{Y})}$$

$$= \frac{1}{n}\log\frac{p(X_1)p(X_2)\cdots p(X_n)\;p(Y_1)p(Y_2)\cdots p(Y_n)}{p(X_1,Y_1)p(X_2,Y_2)\cdots p(X_n,Y_n)}$$

$$= \frac{1}{n}\sum_{\ell=1}^{n}\log\frac{p(X_\ell)p(Y_\ell)}{p(X_\ell,Y_\ell)}$$

$$\rightarrow \; E_{X,Y}\log\frac{p(X)p(Y)}{p(X,Y)}$$ in probability by WLLN

$$= -I(X;Y)$$

3.3 Let X_ℓ be fraction that is kept at each stage, i.e.,

$$X_\ell = \begin{cases} 2/3 & \text{w.p. } 3/4 \\ 3/5 & \text{w.p. } 1/4 \end{cases}$$

Then size after step n is

$$S_n = X_1 \cdot X_2 \cdot \cdots \cdot X_n$$

We want the limit as $n \to \infty$ of

$$\frac{1}{n} \log S_n = \frac{1}{n} [\log X_1 + \cdots + \log X_n]$$

$$\to E[\log X] \quad \text{in probability}$$

$$= \frac{3}{4} \log \frac{2}{3} + \frac{1}{4} \log \frac{3}{5}$$

3.4

a) By Weak LLN (WLLN)

$$P\left[\left|\frac{1}{n}\sum_{l=1}^{n}\log p(X_l) - E[\log p(X)]\right| > \varepsilon\right] \to 0$$

as $n \to \infty$

Since $E \log p(X) = -H(X)$

$$P\left[\left|-\frac{1}{n}\log p(\underline{X}) - H(X)\right| \leq \varepsilon\right] \to 1 \quad \text{as}$$

$n \to \infty$

(b) By WLLN $P[B^n] \to 1$ as $n \to \infty$.

$$P[\underline{X} \in A^n \cap B^n] = 1 - P[\overline{A^n \cap B^n}]$$

$$= 1 - P[\overline{A^n} \cup \overline{B^n}]$$

$$\geq 1 - P[\overline{A^n}] - P[\overline{B^n}]$$

But $P[\overline{A^n}] \to 0$ as $n \to \infty$

$P[\overline{B^n}] \to 0$ as $n \to \infty$

and $P[\underline{X} \in A^n \cap B^n] \leq 1$,

so by squeeze (or sandwich) thm,

$$P[\underline{X} \in A^n \cap B^n] \to 1$$

(c) $|A^n \cap B^n| \le |A^n| \le 2^{n(H(X)+\epsilon)}$

d) Let N be large enough that for all $n > N$, $P[A^n \cap B^n] \ge \frac{1}{2}$

Then
$$\frac{1}{2} \le P[A^n \cap B^n]$$
$$= \sum_{\underline{x} \in A^n \cap B^n} p(\underline{x})$$
$$\le \sum_{x \in A^n \cap B^n} 2^{n(H(X)-\epsilon)}$$
$$= |A^n \cap B^n| \, 2^{n(H(X)-\epsilon)}$$

$$\Rightarrow \quad |A^n \cap B^n| \ge \frac{1}{2} 2^{n(H(X)-\epsilon)}$$

for all $n > N$.

3.5

(a) $1 \geq P[C_n(t)]$

$$= \sum_{\underline{x} \in C_n(t)} p(\underline{x})$$

$$\geq \sum_{\underline{x} \in C_n(t)} 2^{-nt}$$

$$= |C_n(t)| 2^{-nt}$$

$$\Rightarrow |C_n(t)| \leq 2^{nt}$$

(b) From Thm 3.3.1 we know that

$$\frac{1}{n} \log |C_n(t)| > H - \delta'$$

for any $\delta' > 0$. So $t \geq H$ is necessary.

Now, say $t > H$, say $t = H + \delta$, $\delta > 0$.

Then $A_\varepsilon^{(n)} \subseteq C_n(t)$ for all $0 < \varepsilon < \delta$.

$\Rightarrow P[C_n(t)] \geq P[A_\varepsilon^{(n)}] \geq 1 - \varepsilon$ for n large enough

Since $\delta > \varepsilon > 0$ is arbitrary,

$\lim P[C_n(t)] \to 1$.

Since $\delta > 0$ is arbitrary, $t > H$ is sufficient.

3.6 X must be non-negative, or the limit is not well defined.

If $P[X=0]>0$ then the limit is zero since almost surely, one X_ℓ will be 0 in the product.

So assume the PMF of X allows only strictly +ve values, and say

$$\lim_{n\to\infty} [p(X_1,\dots,X_n)]^{1/n} \to L \quad \text{in probability.}$$

Then, by the continuous mapping theorem, this is equivalent to

$$\lim_{n\to\infty} \log\{[p(X_1,\dots,X_n)]^{1/n}\} \to \log\{L\} \quad \text{in probability}$$

But $\lim_{n\to\infty} \log\{[p(X_1,\dots,X_n)]^{1/n}\}$

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{\ell=1}^{n} \log p(X_\ell) \to E \log p(X) \quad \text{in probability}$$

$$= -H(X)$$

$$\Rightarrow L = 2^{-H(X)}$$

3.8. Like Problem 3.6, we study

$$\log (X_1 \cdots X_n)^{1/n}$$

$$= \frac{1}{n} \sum_{l=1}^{n} \log X_l \to E \log X \quad \text{in probability}$$

$$\Rightarrow (X_1 \cdots X_n)^{1/n} \to 2^{E \log X} \quad \text{in probability}$$

with $E \log X = \frac{1}{2} \log 1 + \frac{1}{4} \log 2 + \frac{1}{4} \log 3$

$$= \frac{1}{4} + \frac{1}{4} \log_2 (3)$$

$\underline{3.9}$

(a) $-\frac{1}{n}\log q(X_1, \dots, X_n)$

$$= -\frac{1}{n}\sum_{l=1}^{n}\log(q(X_l))$$

$$\to -E_{\sim p}\log q(X) \quad \text{in probability}$$

(b) $+\frac{1}{n}\log \frac{q(X_1, \dots, X_n)}{p(X_1, \dots, X_n)}$

$$= +\frac{1}{n}\sum_{l=1}^{n}\log\frac{\cancel{p}q(X_l)}{p(X_l)}$$

$$\to +E_{\sim p}\log\frac{q(X)}{p(X)} \quad \text{in probability}$$

$$= -D(p\|q)$$

3.10 $\quad V_n = \prod_{\ell=1}^{n} X_\ell$

Assume $\lim_{n\to\infty} V_n^{1/n} = L$ in probability

Then $\lim_{n\to\infty} \log V_n^{1/n}$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{\ell=1}^{n} \log X_n$$

$$= E \log X \quad \text{in probability}$$

$$= \log L$$

$$\Rightarrow L = e^{E \log X}$$ (assuming natural log is used)

Now $E \log X$

$$= \int_0^1 \log x \, dx = x \log x - x \Big|_0^1$$

$$= -1$$

$$\Rightarrow L = e^{-1} \simeq 0.36788$$

By comparison $[E V_n]^{1/n} = \left(E[X_1] \cdots E[X_n]\right)^{1/n}$

$$= \left(\tfrac{1}{2} \cdots \tfrac{1}{2}\right)^{1/n}$$

$$= \tfrac{1}{2}$$