Sta457 Time Series Analysis
Midterm Exam Practice

Name(Print!):______________________________ Student ID:______________________________

- Do not turn the page until told to do so.
- Do not sit directly next to another student.
- Take note of your exam number, which is written in the top right corner of this page. While the exam is being given, you will be asked to sign an attendance sheet, on the row indexed by your exam number.
- This is a closed book examination. You should have your hand calculator (non-programmable) and you may use one (double-sided) sheet of formulas. You should have no other written material with you during the exam.
- If a question asks you do some calculations, you must show your work to receive full credit. In particular, if you are basing your calculations on a formula, write down that formula before you substitute numbers into it.
- If a later part of a question depends on an earlier part, the later part will be graded conditionally on how you answered the earlier part, so that a mistake on the earlier part will not cost you points on the later part (unless perhaps the previous mistake was absolutely horrendous). If you can't work out the actual answer to an earlier part, put down your best guess and proceed.
- Write your answers in the spaces provided. If you do not have enough room to show all your work in the space provided, use the back of a nearby page; in such cases write something like "see also the back of page n" and be sure to mark clearly which problem the material on the back of any page refers to. If you pull the pages apart, sign all the pages.
- If you don't understand a question, or are having some other difficulty, see your instructor or TA.

1. a . Find the coefficients ψ_j, $j = 0, 1, 2, \cdots$ in the representation

$$X_t = \sum_{j=0}^{\infty} \psi_j a_{t-j}$$

of the ARMA(1,1) process

$$(1 - 0.5B)X_t = (1 + 0.25B)a_t,$$

where (a_t) is a white noise process with variance σ^2.

b . Find the first two ACF's $\rho(1)$ and $\rho(2)$ for X_t defined in (a) above.

2. Suppose that we have the following non-stationary times series model :

$$Z_t = 2\frac{t}{n} + W_t, \quad \text{for } t = 1, 2, \cdots, n,$$

where W_t is an AR(1) process $W_t = .5W_{t-1} + a_t$, where (a_t) are i.i.d. standard normal random variables.

a . Calculate $\text{Corr}(Z_1, Z_2)$ and $\text{Corr}(Z_{n/2}, Z_{n/2+1})$. Are they equal?

b . Calculate the first order **sample** ACF $\hat{\rho}(1)$ for (Z_t) by assuming that n goes to infinity. Is it the same as the the first order ACF of (W_t)?

c . If we want to remove the non-stationary trend $2\frac{t}{n}$ from the time series. One way we learned is to do a regression of Z_t on $\frac{t}{n}$ and subtract the fitted values of the regression from Z_t. More specifically, we do the following regression

$$Z_t = a + b\frac{t}{n} + e_t,$$

Let $\hat{b}$ be the estimate of b. Find the central limit theorem of $\hat{b}$. In other words, find σ such that

$$\sqrt{n}(\hat{b} - 2) \to N(0, \sigma^2).$$

PS: You can use an extended version of the CLT for Sta457 here; that is, if $\{X_i\}$ is a time series (not necessarily stationary) such that $E(X_i)$=0 and $cov(X_i, X_{i+k}) \to 0$ as $k \to \infty$ for any i, then $\sum_{i=1}^{n} X_i/\sqrt{n}$ converges to a normal distribution with mean 0 and variance $E[(\sum_{i=1}^{n} X_i)^2]/n$.

d . Let $Y_t = Z_t - Z_{t-1}$. Prove that Y_t, $t = 2, 3, \cdots, n$ is a weakly stationary time series. Therefore we can transform Z_t into a stationary time series by taking the first order difference of it.

e . In (d) and (e), I proposed two ways to transform (Z_t) into a stationary time series. Which way do you think is better in practice? Give me your reasons.

f . Now suppose Z_t is a little more complicated and is of the following form

$$Z_t = w_1 \frac{t}{n} + (w_2 + \frac{t}{n}) W_t, \quad \text{for } t = 1, 2, \cdots, n,$$

where W_t is an AR(1) process $W_t = .5W_{t-1} + a_t$. Let $Y_t = Z_t - Z_{t-1}$. Prove that Y_t, $t = 2, 3, \cdots, n$ is **not** a weakly stationary time series.

g . Suppose that you don't know the values of w_1 and w_2. Find a way to transform Z_t in (f) into a stationary time series.

3. . Suppose X_t and Y_t are two MA processes of the following forms

$$\begin{aligned} X_t &= a_t - 0.4a_{t-1} \\ Y_t &= b_t + 0.6b_{t-1}, \end{aligned}$$

where (a_t) are iid standard normal random variables and (b_t) are iid standard normal random variables. Further assume the two processes (a_t) and (b_t) are independent.

a . Let $W_t = X_t^2 Y_t^2$. Find the first order ACF $\rho(1)$ of W_t.

b . Let $Z_t = X_t + Y_t$. Find for Z_t the ACF $\rho(k)$ for all k.

4. Suppose that $\{X_t\}$ follows the MA(1) model $X_t = Z_t + 2Z_{t-1}$, where $Z_t \sim WN(0, 1)$. Suppose $X_1 = 1$ and $X_2 = -0.5$. Find the best forecast of X_3 and the associated mean squared forecast error.