A

MAT 1322 3X S2012 29th of May 19:00 20:20 Prof. C. Rada

MIDTERM TEST 1

Max = 20

SOL: 

Student Number: ______________________________

•Time: 80 min.

• Only basic scientific calculators are permitted (non-graphing, non-programmable, no integration or differentiation capabilities). Notes or books are not permitted.

• Work all problems in the space provided. Use the backs of the pages for rough work if necessary. Do not use any other paper.

• The problems require complete and clearly presented solutions and carry part marks if there is substantial correct work towards the solution.

• There are five questions worth four marks each.

(A)

1. (a) Consider the integral $\int_1^4 \frac{3}{(x-3)^{5/3}}\,dx$. Does it converge or diverge? If it converges, give its value.

Consider $\int_1^3 \frac{3}{(x-3)^{5/3}}\,dx$ and $\int_3^4 \frac{3}{(x-3)^{5/3}}\,dx$.

NOTE: $\int_1^3 3(x-3)^{-5/3}\,dx = \lim_{t\to 3^-} \int_1^t 3(x-3)^{-5/3}\,dx =$

$= \lim_{t\to 3^-} \frac{3(x-3)^{-2/3}}{-\frac{2}{3}}\Big|_1^t = -\frac{9}{2}\lim_{t\to 3^-}\left(\frac{1}{(t-3)^{+2/3}} - \frac{1}{(-2)^{2/3}}\right)$

$= -\infty$, SO DIV.

Hence: $\int_1^4 \frac{3}{(x-3)^{5/3}}\,dx$ is Divergent

(b) Use the Comparison Test to determine if the integral $\int_2^\infty \frac{21+\sin x}{x^3+8x}\,dx$ converges or diverges.

For all $x \geq 2 \Rightarrow -1 \leq \sin x \leq 1 \Rightarrow 20 \leq 21+\sin x \leq 22$

For all $x \geq 2 \Rightarrow x^3+8x \geq x^3$, so $\frac{1}{x^3+8x} \leq \frac{1}{x^3}$.

Hence $\frac{21+\sin x}{x^3+8x} \leq \frac{22}{x^3} = 22\cdot\frac{1}{x^3}$ for all $x \geq 2$.

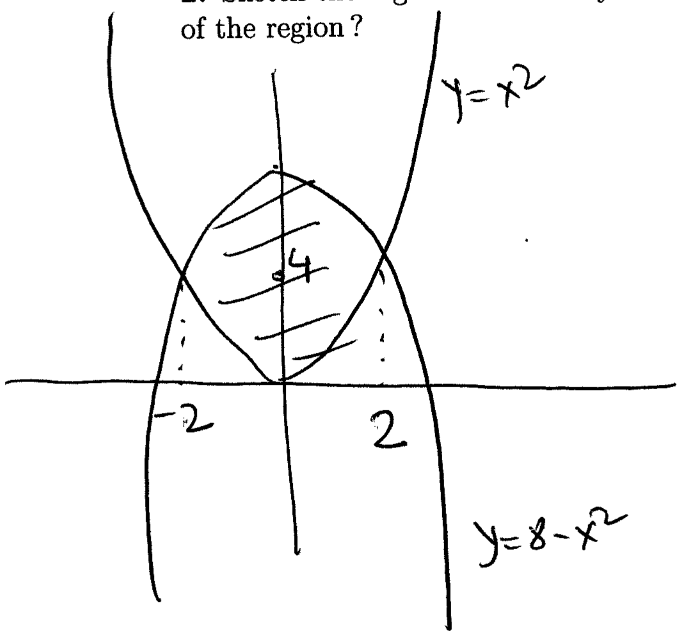
Since $\int_2^\infty \frac{1}{x^3}\,dx$ is convergent $(p = 3 > 1)$, by

Comparison test one gets that $\int_2^\infty \frac{21+\sin x}{x^3+8x}\,dx$

is Convergent

(A)

2. Sketch the region bounded by the curves $y = x^2$ and $y = 8 - x^2$. What is the area of the region?



Cut Points:

$x^2 = 8 - x^2$

$2x^2 = 8 \Rightarrow \quad x^2 = 4 \Rightarrow x = \pm 2$

so : $y = 4$

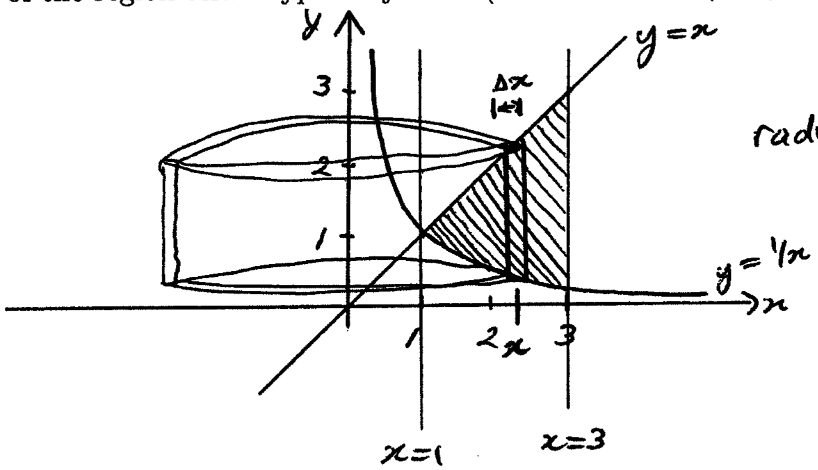
$$A = \int_{-2}^{2} \left(8 - x^2 - x^2\right) dx = \int_{-2}^{2} \left(8 - 2x^2\right) dx$$

$$= \left(8x - 2 \cdot \frac{x^3}{3}\right)\Big|_{-2}^{2} = 16 - \frac{16}{3} - \left[-16 + \frac{16}{3}\right]$$

$$= \frac{64}{3} \approx 21.33$$

A

3. Use the method of cylindrical shells to find the volume of the solid obtained when the region bounded by $y = x$, $y = 1/x$, $x = 1$ and $x = 3$ is rotated around the y-axis. Include a sketch of the region and a typical cylinder (with dimensions) in your solution.



radius of shell is x

height is $x - 1/x$

thickness is Δx

so volume of shell is $2\pi x \left(x - 1/x\right)\Delta x$

and the volume of the solid is

$$V = \int_1^3 2\pi x \left(x - 1/x\right) dx$$

$$= 2\pi \int_1^3 (x^2 - 1)\,dx$$

$$= 2\pi \left(\tfrac{1}{3}x^3 - x \Big|_1^3 \right)$$

$$= 2\pi \left[(9-3) - (1/3 - 1) \right]$$

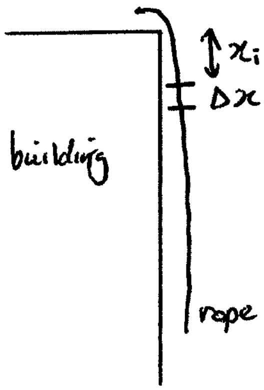
$$= 2\pi \left(7 - 1/3\right)$$

$$= 2\pi \left(20/3\right)$$

$$= \boxed{40\pi/3} \simeq \boxed{41.89}$$

A

4. A heavy rope of length 12 m has a density of 1.5 kg/m and is hanging over the edge of a tall building. How much work is done pulling the rope to the top of the building? The acceleration of gravity is $g = 9.8\ \mathrm{m/s^2}$. Define clearly all the variables that enter into your solution and provide a drawing which shows their meaning.



Chop the rope into lengths Δx m.
Consider the length x_i m below the top of the building

This length of rope has mass $1.5\Delta x$ kg and weight $1.5\, g\, \Delta x = 14.7\Delta x$ N

This length must be lifted x_i m, so the work done on this piece is $W_i = 14.7\, x_i \Delta x$ J

so the total work done is $W \approx \sum_i W_i = \sum_i 14.7 x_i \Delta x$ J

take the limit as $\Delta x \to 0$ to get

$$W = \lim_{\Delta x \to 0} \sum_i 14.7 x_i \Delta x = \int_0^{12} 14.7\, x\, dx$$

$$= 14.7 \int_0^{12} x\, dx$$

$$= 14.7 \left(\tfrac{1}{2} x^2 \Big|_0^{12}\right)$$

$$= 14.7\,(144/2)$$

$$= \boxed{1058.4\ \mathrm{J}}$$

A

5. Give the integral for the average value of the function $f(x) = \arcsin x$ on the interval $[0, 1/2]$ and then evaluate it.

$$f_{ave} = \frac{1}{\frac{1}{2}-0}\int_0^{\frac{1}{2}} \arcsin x\, dx = 2\int_0^{\frac{1}{2}} 1 \cdot \arcsin x\, dx =$$

NOTE: $\left(\begin{array}{l} f'(x)=1 \to f(x)=x \\ g(x)=\arcsin x \to g'(x)=\frac{1}{\sqrt{1-x^2}} \end{array} \right)$; so by Int. by Parts

$$= 2\left[x\arcsin x \Big|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}}\, dx \right] =$$

$$= 2\left[x \cdot \arcsin x \Big|_0^{\frac{1}{2}} + (1-x^2)^{\frac{1}{2}} \Big|_0^{\frac{1}{2}} \right]$$

$$= 2\left[\frac{1}{2}\arcsin\left(\frac{1}{2}\right) - 0 + \sqrt{\frac{3}{4}} - 1 \right]$$

$$= 2\left[\frac{1}{2}\cdot\frac{\pi}{6} + \frac{\sqrt{3}}{2} - 1 \right] = \frac{\pi}{6} + \sqrt{3} - 2 \simeq 0.2556$$

op

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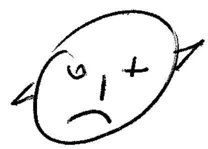
Max = 20

SOL 

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• The problems require complete and clearly presented solutions and carry part marks if there is substantial correct work towards the solution.

• There are five questions worth four marks each.

1. (a) Consider the integral $\int_1^5 \frac{3}{(x-4)^{7/3}}\,dx$. Does it converge or diverge? If it converges, give its value.

Consider $\int_1^4 \frac{3}{(x-4)^{7/3}}\,dx$ AND $\int_4^5 \frac{3}{(x-4)^{7/3}}\,dx$.

Note: $\int_1^4 \frac{3}{(x-4)^{7/3}}\,dx = \lim_{t\to 4^-} \int_1^t \frac{3}{(x-4)^{7/3}}\,dx =$

$= \lim_{t\to 4^-} -\frac{9}{4}(x-4)^{-4/3}\Big|_1^t$ (by Power Rule)

$= \lim_{t\to 4^-} -\frac{9}{4}\left[\frac{1}{(t-4)^{4/3}} - \frac{1}{(-3)^{4/3}}\right] = -\infty \Rightarrow$ D

So: $\int_1^5 \frac{3}{(x-4)^{7/3}}\,dx$ is DIVERGENT

(b) Use the Comparison Test to determine if the integral $\int_1^\infty \frac{4-\cos x}{x^4+3x}\,dx$ converges or diverges.

FOR ALL $x \geq 1$ one has: $-1 \leq \cos x \leq 1$, hence $1 \geq -\cos x \geq -1$

hence $5 \geq 4-\cos x \geq 3$.

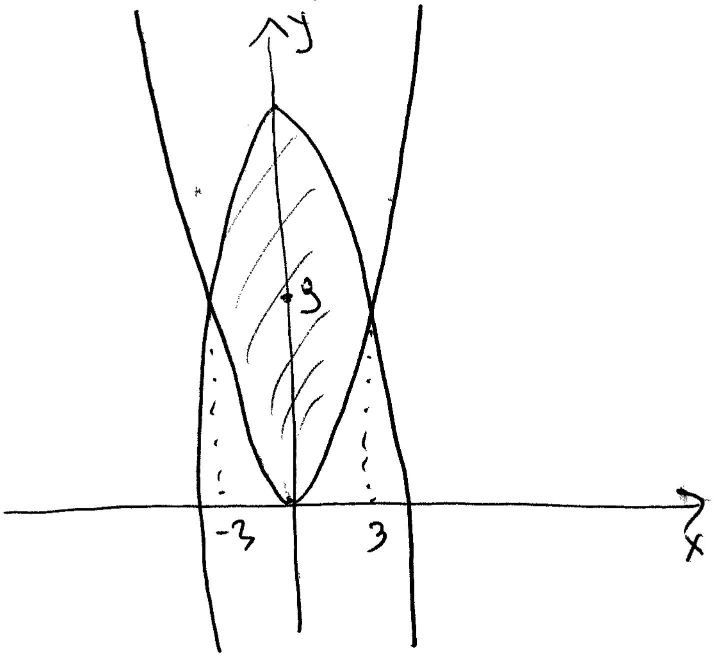
for all $x \geq 1$ one has: $x^4+3x \geq x^4$, so $\frac{1}{x^4+3x} < \frac{1}{x^4}$

Hence: $\frac{4-\cos x}{x^4+3x} \leq \frac{5}{x^4} = 5 \cdot \frac{1}{x^4}$

Since $\int_1^\infty \frac{1}{x^4}\,dx$ is Convergent $(p=4>1)$,

by Comparison Test $\int_1^\infty \frac{4-\cos x}{x^4+3x}\,dx$ is Convergent

2. Sketch the region bounded by the curves $y = x^2$ and $y = 18 - x^2$. What is the area of the region?



Cut POINTS:

$18 - x^2 = x^2 \Rightarrow 18 = 2x^2 \Rightarrow$

$y = x^2 \Rightarrow x = \pm 3$

so $y = 9$

$$A = \int_{-3}^{3} (18 - x^2 - x^2)\,dx = \int_{-3}^{3} 18 - 2x^2\,dx$$

$$= \left(18x - 2 \cdot \frac{x^3}{3}\right)\Bigg|_{-3}^{3} = 18 \cdot 3 - \frac{2}{3} \cdot 3^3 - \left[18(-3) - \frac{2}{3}(-3)^3\right]$$

$$= 54 - 18 - [-54 + 18] = 72$$

(B)

3. Use the method of cylindrical shells to find the volume of the solid obtained when the region bounded by $y = x$, $y = 1/x$, $x = 1$ and $x = 2$ is rotated around the y-axis. Include a sketch of the region and a typical cylinder (with dimensions) in your solution.

diagram as in Version A, ending at $x = 2$

dimensions of shell the same, volume same

so $$V = \int_1^2 2\pi x \,(x - 1/x)\,dx$$

$$= 2\pi \int_1^2 (x^2 - 1)\,dx$$

$$= 2\pi \left(\tfrac{1}{3}x^3 - x \Big|_1^2 \right)$$

$$= 2\pi \left[(8/3 - 2) - (1/3 - 1) \right]$$

$$= 2\pi \,(7/3 - 1)$$

$$= 2\pi \,(4/3)$$

$$= \boxed{8\pi/3} \simeq \boxed{8.3776}$$

(B)

4. A heavy rope of length 10 m has a density of 1.25 kg/m and is hanging over the edge of a tall building. How much work is done pulling the rope to the top of the building? The acceleration of gravity is $g = 9.8 \text{ m/s}^2$. Define clearly all the variables that enter into your solution and provide a drawing which shows their meaning.

diagram as in version A

mass of length is $1.25 \Delta x$ kg

weight of length is $1.25 g \Delta x = 12.25 \Delta x$ N

work is $W \approx \sum_i W_i = \sum_i 12.25 x \Delta x$ J

so $W = \int_0^{10} 12.25 \, x \, dx$

$= 12.25 \left(\frac{1}{2} x^2 \Big|_0^{10} \right)$

$= \boxed{612.5 \text{ J}}$

5. Give the integral for the average value of the function $f(x) = \arctan x$ on the interval $[0, \sqrt{3}]$ and then evaluate it.

$$f_{ave} = \frac{1}{\sqrt{3}-0} \int_0^{\sqrt{3}} \arctan x \, dx = \frac{1}{\sqrt{3}} \int_0^{\sqrt{3}} 1 \cdot \arctan x \, dx =$$

NOTE: $f'(x) = 1 \rightarrow f(x) = x$
$g(x) = \arctan x \rightarrow g'(x) = \frac{1}{1+x^2}$, so By Int. by Parts

$$= \frac{1}{\sqrt{3}} \left[x \cdot \arctan x \Big|_0^{\sqrt{3}} - \int_0^{\sqrt{3}} \frac{x}{1+x^2} \, dx \right] =$$

$$= \frac{1}{\sqrt{3}} \left[x \cdot \arctan x \Big|_0^{\sqrt{3}} - \frac{1}{2} \ln(1+x^2) \Big|_0^{\sqrt{3}} \right]$$

$$= \frac{1}{\sqrt{3}} \left[\sqrt{3} \arctan(\sqrt{3}) - \frac{1}{2} \ln 4 + \underbrace{\frac{1}{2} \ln 1}_{=0} \right]$$

$$= \frac{1}{\sqrt{3}} \left[\sqrt{3} \cdot \frac{\pi}{3} - \ln 2 \right]$$

$$= \frac{\pi}{3} - \frac{\ln 2}{\sqrt{3}} \approx 0.6470$$