Lecture Week 1: ALGEBRAIC FUNCTIONS
(*REVIEW of HIGN SCHOOL MATHEMATICS*)

1.1. Definition of a Function.

A function f from set A to set B (or simply function) is a rule that assigns to each element (or number) x of the domain set A exactly one element called $f(x)$ of the codomain set B.

We can also say that

- function f is an assignment that associates to each element in the set A exactly one element of set B
- function f maps elements from the set A into elements in set B.
- $y = f(x)$ is the image of x under transformation f.
- function f expresses a relationship between two sets of numbers.

We'll be using notation "f from A to B" $f: A \to B$ or $y = f(x)$

x is called independent variable, y is called dependent variable

There are four ways to represent the function: tables, diagrams, formulas, and graphs.
We'll be using notation R to define the set of real numbers.

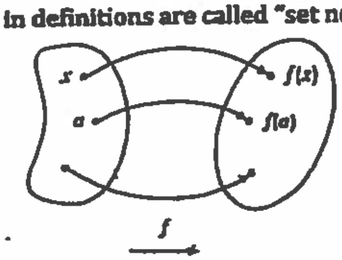
The **domain** of f is the largest subset of R, for which f is defined:

$$Dom(f) = \{x \in R \mid f(x) \text{ is defined}\}$$

The **range** of f is the set of all possible values $y = f(x)$:

$$Ran(f) = \{y \in R \mid y = f(x) \quad for \quad x \in Dom(f)\}$$

Notations used in definitions are called "set notations"; "|..." means "such that", "$\in$" means "contains in"



Machine diagram for a function f

Example:

(a) Consider function $f = \sqrt{x}$. It's defined for all positive numbers. All positive numbers form the set A, which is proper subset of R. In the set notation: $A \subset R$. So, the set A in this example is the domain of $f = \sqrt{x}$. $f(x)$ has domain $[0, \infty)$

(b) Consider function $f = x^2$. It's defined for all real numbers; so all real numbers form the set A. The output of this function is the set B that can contain only positive numbers. The set B is a proper subset of R: $B \subset R$ where the set R is the codomain of f and the set B is the range of $f = x^2$. $f(x)$ has range $[0, \infty)$.

(c) Consider function $f = x^3$. This function is defined for all real numbers and its output is also the set of all real numbers. In this example codomain of f is also its range.

Notations used in example are called "interval notations".

Example: **Express Domain and Range of** $f(x)=\dfrac{1}{\sqrt{x^2-1}}$ **in interval notation.**

$$|x| = \begin{cases} x, & x>0 \\ -x, & x<0 \end{cases}$$

$x^2-1>0$

$x^2>1$

$\sqrt{x^2}>\sqrt{1}$

$|x|>1$

if $x>0$ then $|x|>1 \Rightarrow x>1$

$\begin{cases} x>0 \\ x>1 \end{cases}$  $x\in(0,\infty)\cap(1,\infty) \Rightarrow x\in(1,\infty)$

if $x<0$ then $|x|>1 \Rightarrow -x>1 \Rightarrow x<-1$

$\begin{cases} x<0 \\ x<-1 \end{cases}$  $x\in(-\infty,-1)\cap(-\infty,0) \Rightarrow x\in(-\infty,-$

Conclusion: $\operatorname{dom} f(x) := (1,\infty)\cup(-\infty,-1)$

$\operatorname{Ran} f(x) := (0,\infty)$

1.2. Equality between functions.

Two functions are equal iff:

a) $f(x)=g(x)$ **for all** x **in the domain of** f**;**

b) $f(x)$ *and* $g(x)$ **defined on the same domain.**

Example: **Whether or not** $f(x)=\dfrac{x^2-1}{x-1}$ **is equal to** $g(x)=x+1$ **?**

$$f(x)=\frac{x^2-1}{x-1}=\frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}}=x+1$$

$\operatorname{Dom} f(x)=\{x\in R \mid x\neq 1\}$

$g(x)=x+1$

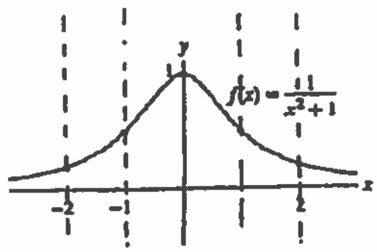
$\operatorname{Dom} g(x)=\{x\in R\}$

Conclusion: $f(x)\neq g(x)$ (b) fails !

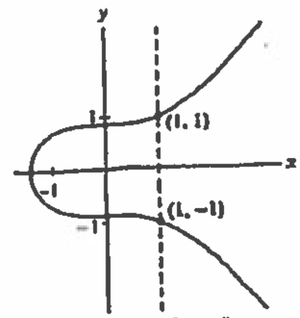
1.3. Vertical Line Test

A curve in the XY plane is the graph of a function Iff no vertical line intercepts the curve more than once.

Function.



Not a Function.

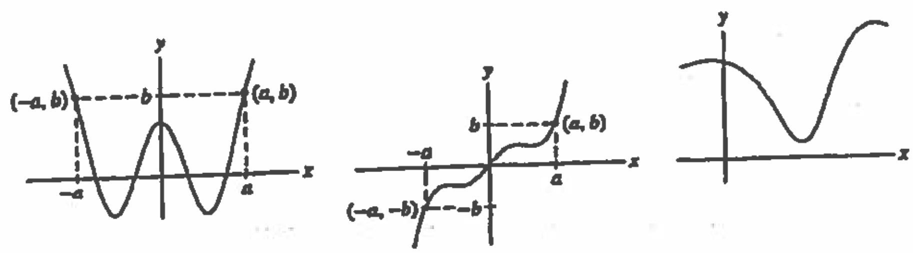


Graph of $4y^2 - x^3 = 3$. This graph fails the Vertical Line Test, so it is not the graph of a function.

1.4 Even and Odd Functions.

A function f is <u>even</u> if $f(-x) = f(x)$ for all x in the domain of $f(x)$.
Graphs of even functions are symmetric with respect to y-axis.
Example: $f(x) = x^2 - 3$ is the even function. Check $f(-x) = (-x)^2 - 3 = x^2 - 3 = f(x)$

A function f is <u>odd</u> if $f(-x) = -f(x)$ for all x in the domain of $f(x)$.
Graphs of odd functions are symmetric with respect to the origin.
Example: $f(x) = x^3$ is the odd function. Check $f(-x) = (-x)^3 = -x^3 = -f(x)$



(A) Even function: $f(-x) = f(x)$ Graph is symmetric about the y-axis.

(B) Odd function: $f(-x) = -f(x)$ Graph is symmetric about the origin.

(C) Neither even nor odd

A function <u>is neither even nor odd</u> if $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$

Example : $f(x) = \dfrac{x}{x^3+2}$ is neither even nor odd function.

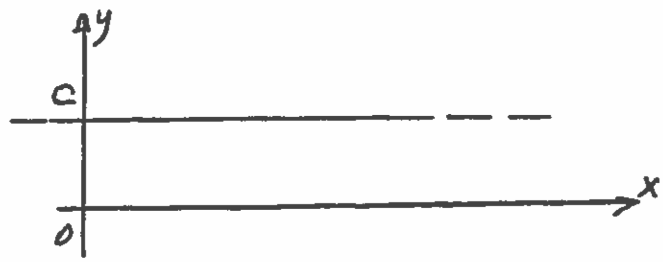
Check; $f(-x) = \dfrac{-x}{(-x)^3+2} = \dfrac{-x}{-x^3+2} \neq f(x)$;

$$f(-x) = \frac{-x}{(-x)^3+2} = \frac{x}{x^3-2} \neq -f(x)$$

1.5 A Library of Algebraic Functions ($f(x) = y$).

1.5.1. Constant Functions

$$y = C$$



Dom $f(x) := \{x \in R\}$,

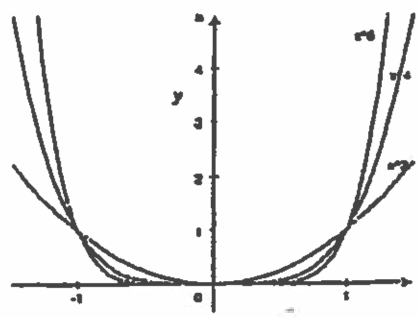
Ran $f(x) := \{y \in R | y = C\}$

1.5.2. Power Functions

$$y = x^p$$

a) *Even Power Functions* (p=2n, n=1,2,3...)

$$y = x^{2n}$$

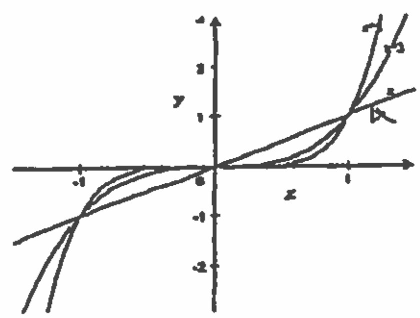


Dom $f(x) := \{x \in R\}$,

Ran $f(x) := \{y \in R | y \geq 0\}$

b) *Odd Power Functions* (p=2n-1,n=1,2,3...)

$$y = x^{2n-1}$$

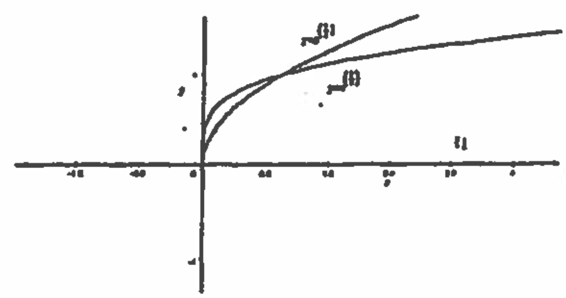


Dom $f(x) := \{x \in R\}$,

Ran $f(x) := \{y \in R\}$

c) *Even n-th Root Functions* ($p = \dfrac{1}{2n}$, n=1,2,3...)

$$y = x^{\frac{1}{2n}}$$

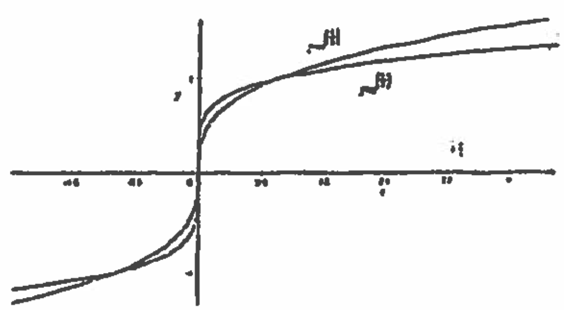


Dom $f(x) := \{x \in R | x \geq 0\}$,

Ran $f(x) := \{y \in R | y \geq 0\}$

d) *Odd n-th Root Functions* ($p = \dfrac{1}{2n-1}$, n=1,2,3...)

$$y = x^{\frac{1}{2n-1}}$$

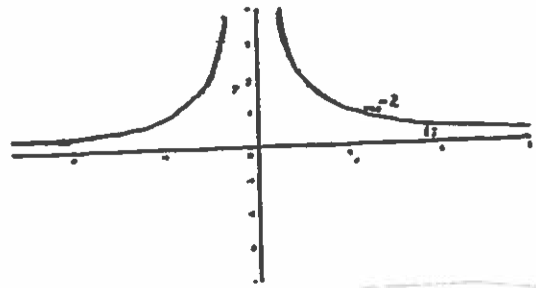


Dom $f(x) := \{x \in R\}$,

Ran $f(x) := \{y \in R\}$

e) ***Negative Even Power Functions*** ($p=-2n$, n=1,2,3...)

$$\underline{y=x^{-2}}$$

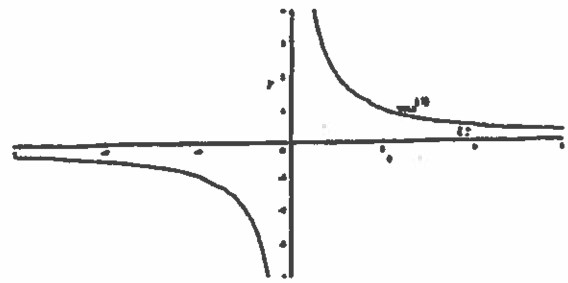


Dom $f(x) := \{x\in R \mid x\neq 0\}$

Ran $f(x) := \{y\in R \mid y>0\}$

f) ***Negative Odd Power Functions*** ($p=-2n+1$, n=1,2,3...)

$$\underline{y=x^{-1}}$$



Dom $f(x) := \{x\in R \mid x\neq 0\}$,

Ran $f(x) := \{y\in R \mid y\neq 0\}$

1.5.3 <u>Polynomial Functions</u>.

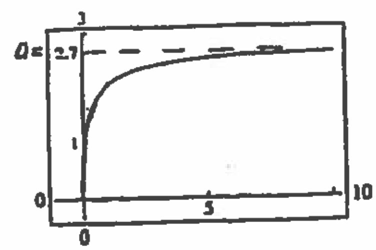
$$\underline{f(x)=a_n x^n + a_{n-1}x^{n-1}+\ldots+a_1 x + a_0} \quad \text{where } \{a_n, a_{n-1}, \ldots a_0\}\in R$$

The degree of a polynomial function equals to the degree of leading term of corresponding polynomial: for n=1 we have linear function.

1.5.4. <u>Rational Functions.</u>

A rational function is any function of the type $f(x)=\dfrac{P_n(x)}{Q_m(x)}$ where $Q_m(x)$ is not the zero polynomial. The Domain of $f(x)$ is the set of all x such that $Q_m(x)\neq 0$.

Example. Monod Function $y(x)=a\dfrac{x}{b+x}$



$n=2 \quad f(x)=a_2x^2 + a_1x+a_0$

$n=1 \quad f(x) = a_1 x + a_0$

$n=0 \quad f(x) = a_0$

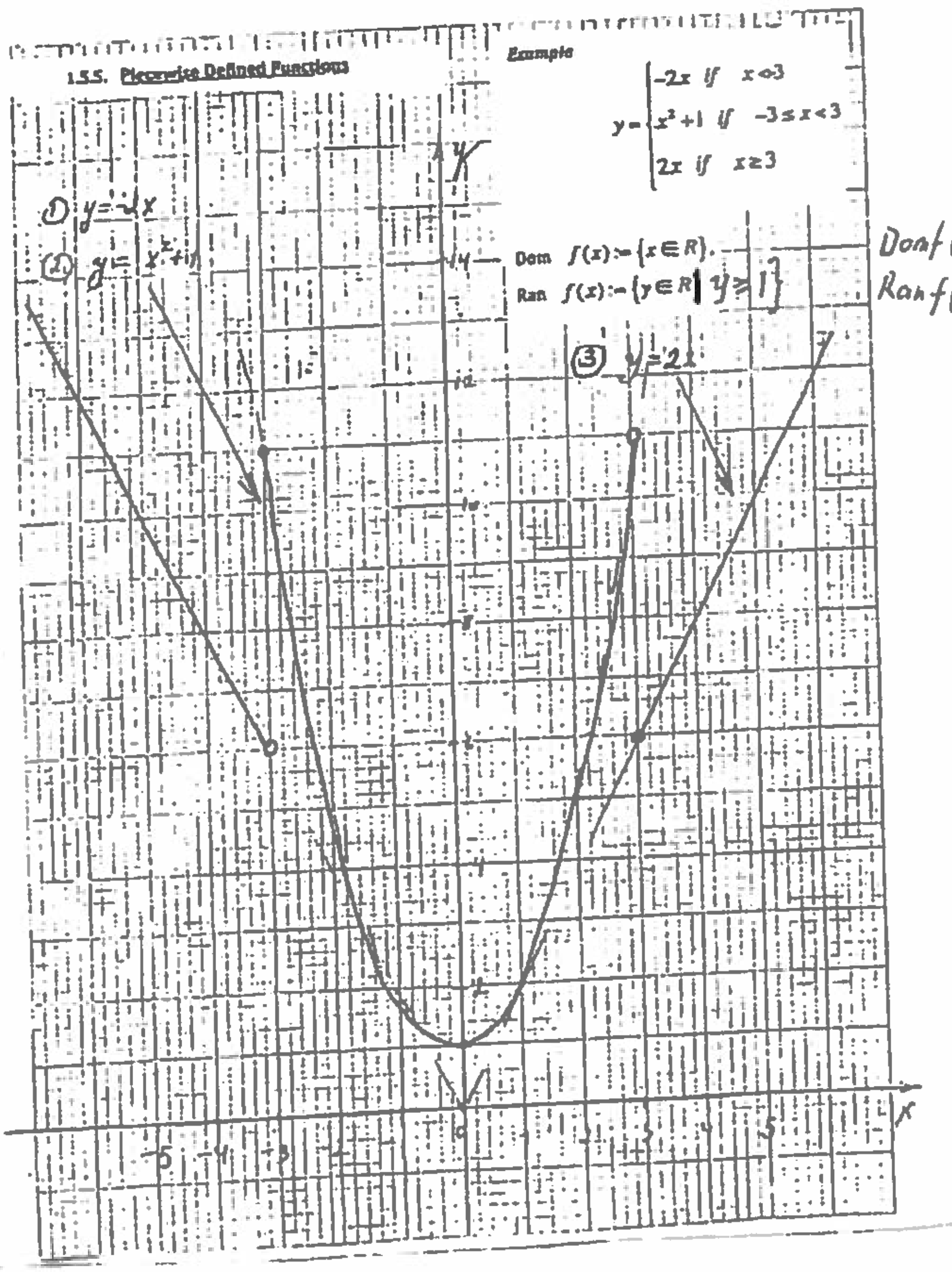
1.5.5. Piecewise Defined Functions

Example

$$y = \begin{cases} -2x \ \ if \ \ x < -3 \\ x^2+1 \ \ if \ \ -3 \le x < 3 \\ 2x \ \ if \ \ x \ge 3 \end{cases}$$

Dom $f(x) := \{x \in R\}$,

Ran $f(x) := \{y \in R \mid y \geq 1\}$

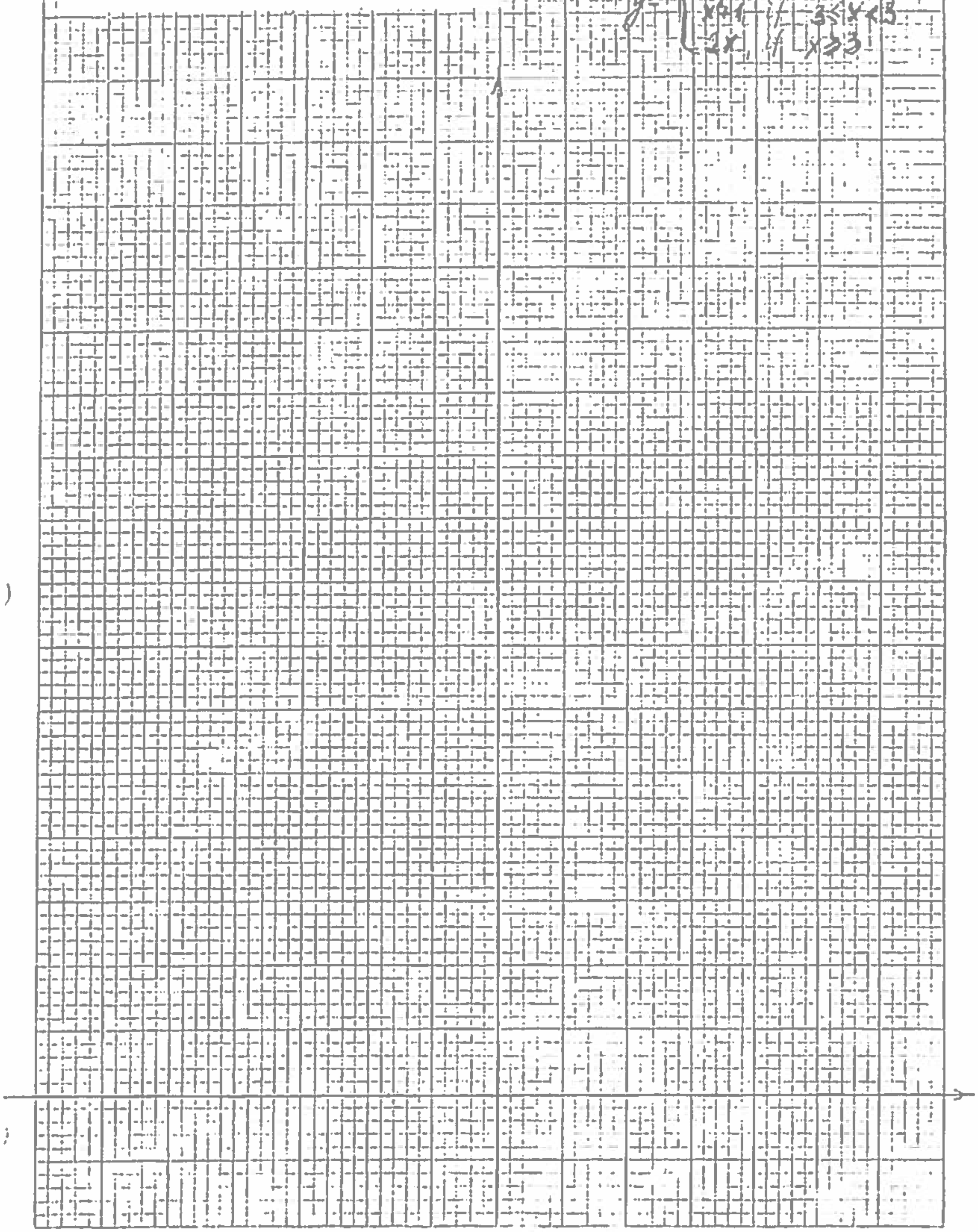


Dom $f(x) := (-\infty, \infty)$

Ran $f(x) := [1, \infty)$

1.5.5. Piecewise Defined Functions

$$y = \begin{cases} -2x, & \text{if } x < -3 \\ x+1, & \text{if } -3 \le x < 3 \\ 2x, & \text{if } x \ge 3 \end{cases}$$



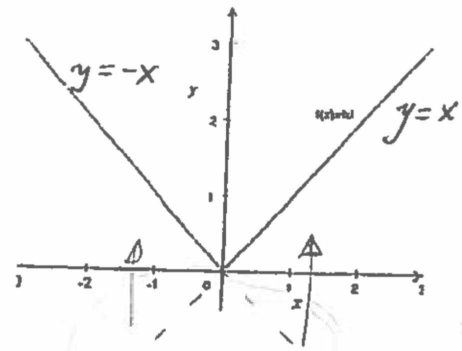
1.5.6. **Absolute Value Functions.**

Example

$$y = |x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

Dom $f(x) := \{x \in R\}$,

Ran $f(x) := \{y \in R | y \geq 0\}$

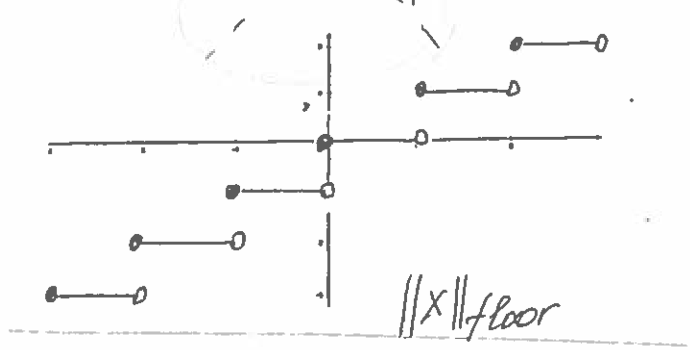


1.5.7 **The Floor and Ceiling Functions**

$$f(x) = \|x\|$$

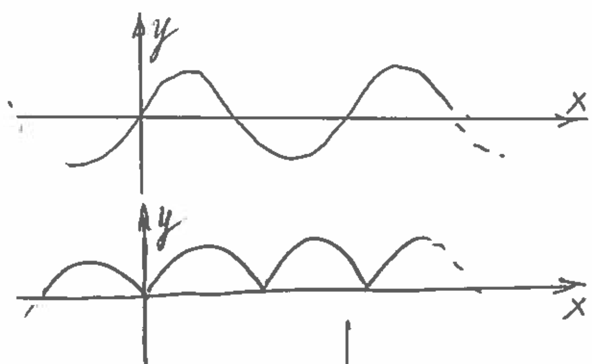
Dom $f(x) := \{x \in R\}$,

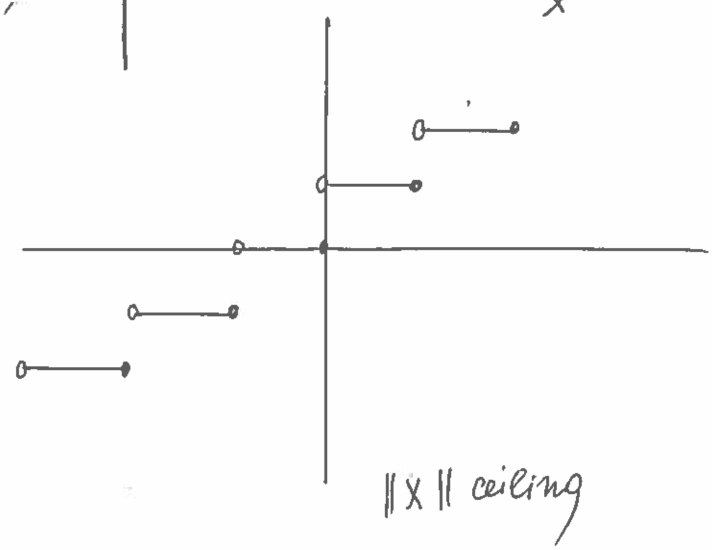
Ran $f(x) := \{y \in Z\}$



$y = \sin x$

$y = |\sin x|$





1.6 Operations with Functions.

Let f and g be functions with domains A and B respectively, then

$(f \pm g)(x) = f(x) \pm g(x)$, $\quad Dom(f+g) := A \cap B$

$(fg)(x) = f(x)g(x)$, $\quad Dom(fg) := A \cap B$

$\dfrac{f}{g}(x) = \dfrac{f(x)}{g(x)}$, $\quad Dom\left(\dfrac{f}{g}\right) := \{x \in A \cap B \mid g(x) \neq 0\}$

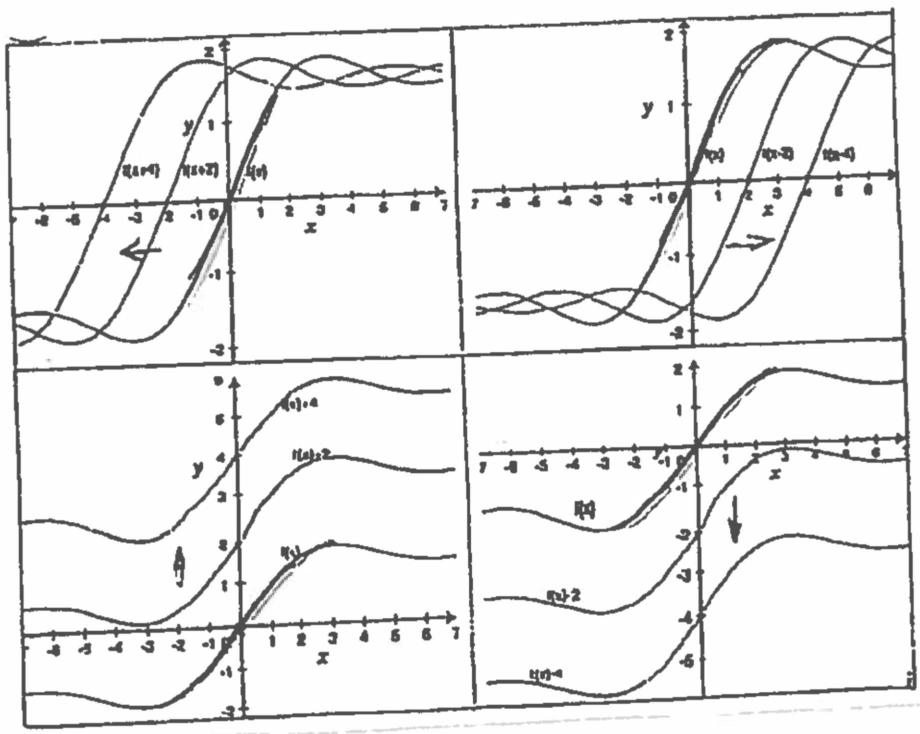
1.6.1 Transformations

a) *Vertical and Horizontal Shifts.*

Suppose $c > 0,\ c \in R$ then to get the graph of

1. $y = f(x) + c$,shift the graph of $y = f(x)$ c units upward
2. $y = f(x) - c$,shift the graph of $y = f(x)$ c units downward
3. $y = f(x - c)$,shift the graph of $y = f(x)$ c units to the right
4. $y = f(x + c)$,shift the graph of $y = f(x)$ c units to the left

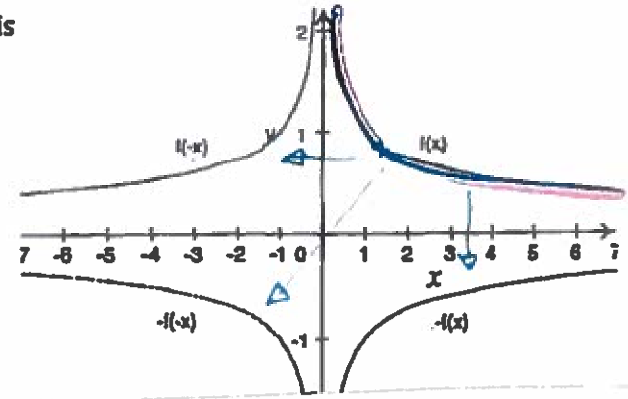
4.
3.
1.
2.



b) Reflecting

Suppose $c>1,\ c\in R$ then to get the graph of

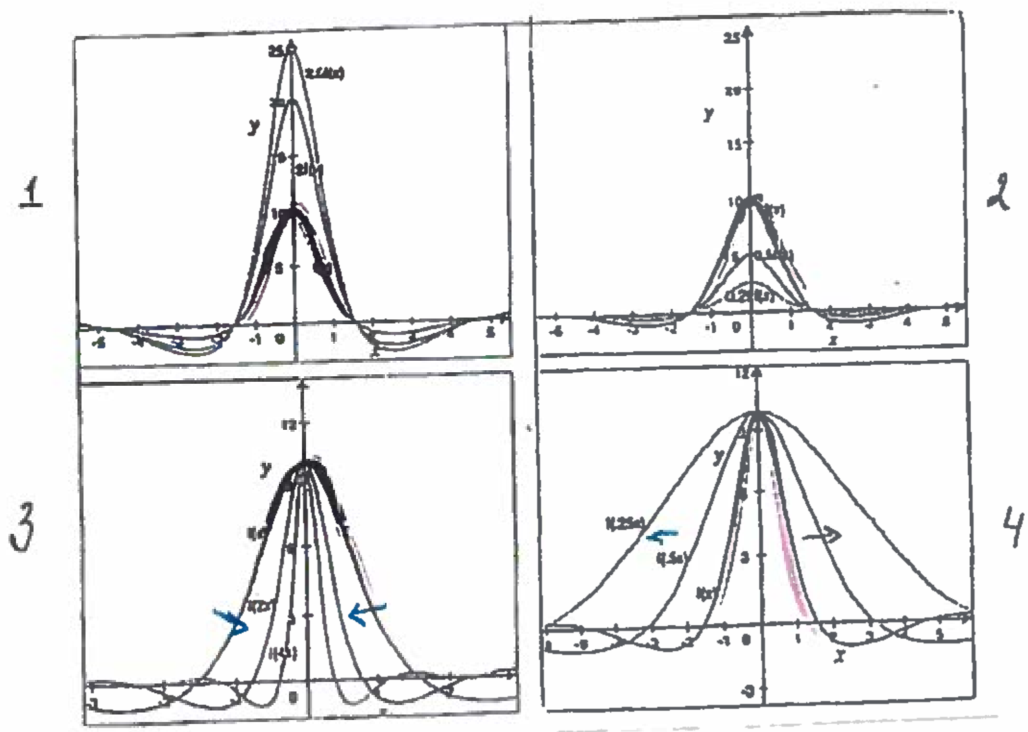
1. $y=-f(x)$, reflect the graph of $y=f(x)$ about the x-axis
2. $y=f(-x)$, reflect the graph of $y=f(x)$ about the y-axis



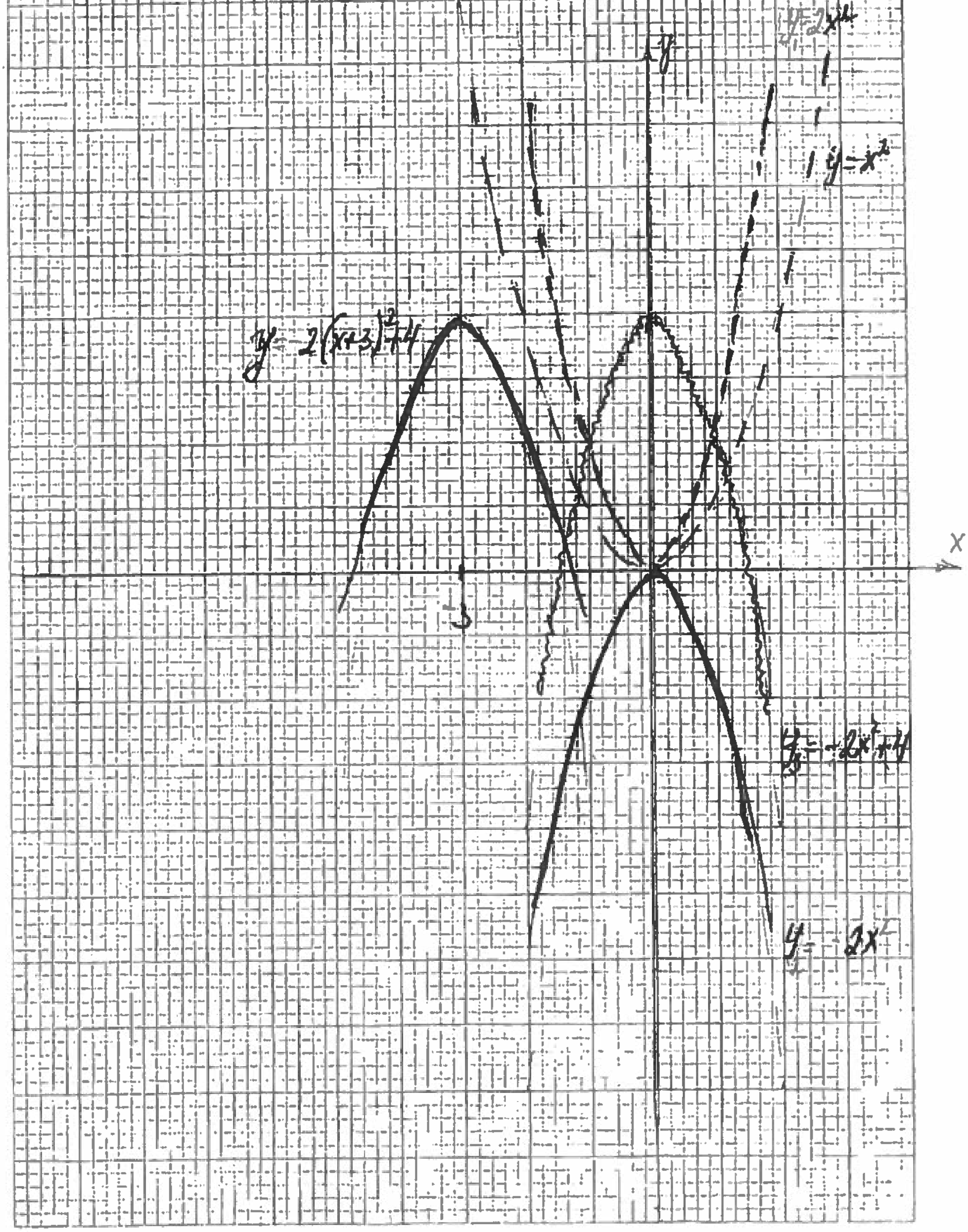
c) Vertical and Horizontal Stretching and Compression.

Suppose $c>1,\ c\in R$ then to get the graph of

1. $y=cf(x)$, stretch the graph of $y=f(x)$ vertically by a factor c
2. $y=\dfrac{1}{c}f(x)$, compress the graph of $y=f(x)$ vertically by the factor c
3. $y=f(cx)$, compress the graph of $y=f(x)$ horizontally by the factor of c
4. $y=f\left(\dfrac{x}{c}\right)$, stretch the graph of $y=f(x)$ horizontally by a factor c



Use the transformation to graph $y=-2(x+3)^2+4$



1.6.2. Composition of Functions

For two functions $f(x)$ *and* $g(x)$ composite function is defined by $f[g(x)] = f \circ g$.

Let D_f *and* D_g be domains of $f(x)$ *and* $g(x)$ respectively, then the domain of $f[g(x)] = f \circ g$ is the set of all x in the domain of g such that $g(x)$ *in* the domain of f: $Dom(f \circ g) := \{x \in D_g | g(x) \in D_f\}$

Example:

Let $f(x) = x^2$ *and* $g(x) = \sqrt{x-3}$ Find $f \circ g$, $g \circ f$, $f \circ f$, $g \circ g$ and their domains.

Answer;

$$f \circ g = f[g] = g^2(x) = (\sqrt{x-3})^2 = x-3 \qquad Dom(f \circ g) := \{x \in R | x \geq 3\}$$

$$g \circ g = g[g] = \sqrt{g-3} = \sqrt{\sqrt{x-3}-3} \qquad Dom(g \circ g) := \{x \in R | x \geq 12\}$$

$$g \circ f = g[f] = \sqrt{f-3} = \sqrt{x^2-3} \qquad Dom(g \circ f) := \{x \in R | x \leq -\sqrt{3}\} \cup \{x \in R | x \geq \sqrt{3}\}$$

$$f \circ f = f[f] = f^2 = x^4 \qquad Dom(f \circ f) := \{x \in R\}$$

We also can apply the rule of composition to more than two functions. For example, the composition of three functions f, g *and* h is defined as $(f \circ g \circ h)(x) = f[g(h(x))]$

Example

Let $f(x) = \frac{1}{x}$, $g(x) = x^3$ *and* $h(x) = x^2 + 2$. Find $(f \circ g \circ h)(x) = f[g(h(x))]$

$$f \circ g \circ h = f[g(h)] = \frac{1}{g} = \frac{1}{(x^2+2)^3}$$

$$g(h) = h^3 = (x^2+2)^3$$

$f(x)$ is one to one if for all a & b in the Dom $f(x)$

$$a \neq b \Rightarrow f(a) \neq f(b)$$

"not A" => "not B"

Contrapositive statement: $f(a) = f(b) \Rightarrow a = b$

$A \Rightarrow B$ "if A then B"

Definition:

(1-1) $\boxed{\text{f(x) is one-to-one if for all a \& b in the Dom f(x)} \quad a \neq b \Rightarrow f(a) \neq f(b)}$

(if A is true, then B is also true)

(1-1) Contrapositive statement: $f(a) = f(b) \Rightarrow a = b$

"if A is true" then "B is also true"

Which is definition of 1-1 func

Ex: Show that $f(x) = 2x + 5$ is 1-1

Assume that $f(a) = f(b)$

$$\cancel{2}a + \cancel{5} = \cancel{2}b + \cancel{5}$$

$a = b$ — conclusion: $f(a) = f(b)$ iff $a = b$

Ex: Whether or not $f(x) = 2x^2 + 5$ is 1-1?

Assume that $f(a) = f(b)$

$$2a^2 + \cancel{5} = 2b^2 + \cancel{5}$$

$$a^2 = b^2$$

$a^2 - b^2 = (a-b)(a+b) = 0$

$a = b$

$a = -b$

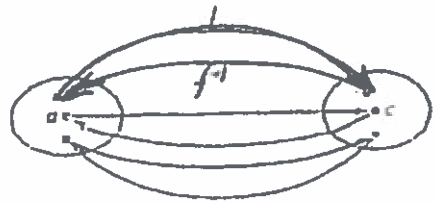
$a = \pm b$ — conclusion:

when $a \neq b \Rightarrow f(a) = f(b)$

So $f(x)$ is not 1-1.

1.8 Inverse Functions.

Let f be a one-to one function with domain A and range B. Then its inverse function f^{-1} has domain B and range A and defined by $f^{-1}(y) = x \Leftrightarrow f(x) = y$



Example

Suppose for a given one-to one function f, $f(1) = 5$, $f(3) = 7$ *and* $f(8) = -10$.

Find $f^{-1}(5)$, $f^{-1}(7)$ *and* $f^{-1}(-10)$.

$f^{-1}(5) = 1$, $f^{-1}(7) = 3$, $f^{-1}(-10) = 8$

! $f^{-1} \neq (f)^{-1}$ or $f^{-1}(x) \neq [f(x)]^{-1}$

1.8.1 Property of Inverse Functions.

Let f be a one-to one function with domain A and range B. The following cancelation properties indicate that f^{-1} is the inverse of f and vice versa.

$f^{-1}[f(x)] = x$ *for all* x *in* A

$f[f^{-1}(x)] = x$ *for all* x *in* B

inverse $f^{-1} \neq \frac{1}{f}$

Example:

Show that $f(x) = x^3$ *and* $g(x) = \sqrt[3]{x}$ are inverses of each other.

reciprocal $(f)^{-1} = \frac{1}{f}$

Solution.

To show that these functions are inverses of each other we use the property on inverse functions. Note that domain and range of both functions is R.

We have $g[f(x)] = g[x^3] = \sqrt[3]{x^3} = x$, $f[g(x)] = f[x^{1/3}] = (\sqrt[3]{x})^3 = x$, so by the property of inverse functions f *and* g are inverses of each other.

$f \circ f^{-1} = x$ $f^{-1} \circ f = x$

1.8.2 Finding the Inverse Function of one-to one Rational Functions.

1. Write $y = f(x)$
2. Solve the equation for x in terms of y
3. Interchange x and y. The resulting function is $y = f^{-1}(x)$

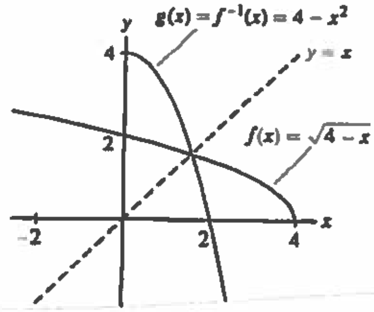
Example:

Find the inverse function of $f(x) = \sqrt{4-x}$ and sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$.

1. $y = \sqrt{4-x} = f(x)$

2. $y^2 = 4 - x$
   $x = 4 - y^2$

3. $y = 4 - x^2 = f^{-1}(x)$

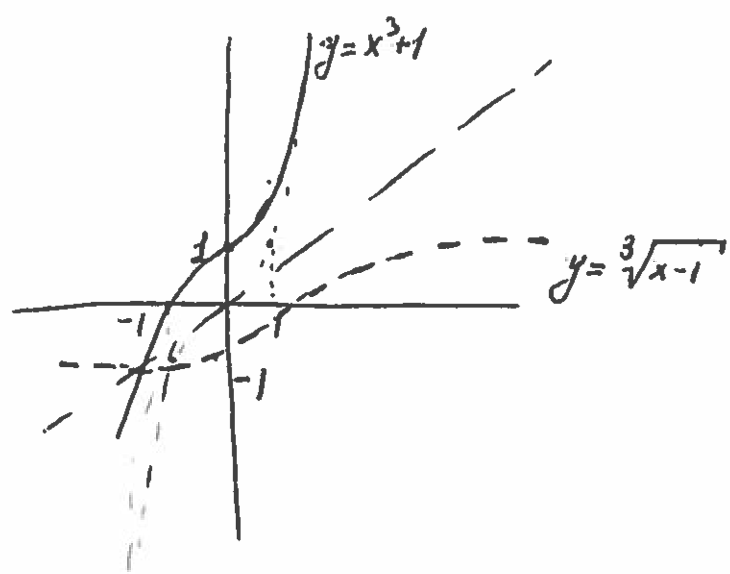


In order to sketch the graph, we observe that $Dom(f(x)) := \{x \in R | x \le 4\}$ and $Ran(f(x)) := \{y \in R | y \ge 0\}$ and the $Ran(f^{-1}(x)) := \{y \in R | x \le 4\}$ and $Dom(f^{-1}(x)) := \{x \in R | x \ge 0\}$. Graphs of $f(x) = \sqrt{4-x}$ and $y = 4 - x^2$ are reflections of each other in $y = x$

Example:

Find the inverse function of $f(x) = x^3 + 1$ and sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$.

1. $y = x^3 + 1 = f(x)$
2. $x^3 = y - 1$
   $x = \sqrt[3]{y-1}$
3. $y = \sqrt[3]{x-1} = f^{-1}(x)$



Examples

$f: A \to B$ (maps elements from set A into set B).

"f from A to B."

$f(x) = x^3$, $f: R \to R$ (from real to real).

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a) Describe a constant function: $f: R \to R$

(I) – any $f(x) = 5$, $f(x) = 2.7$ etc. . . .

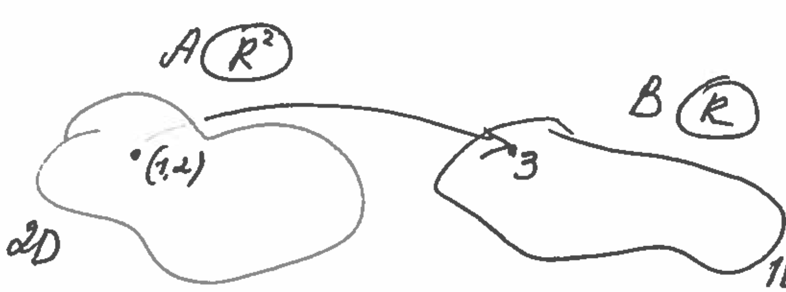
b) $f: R \to [2, 4]$ (set of numbers between 2 and 4).

– any $f(x) = 3$, $f(x) = 2.7$, . . .

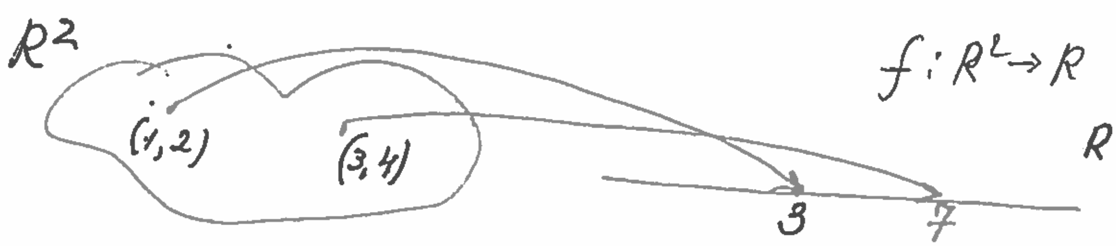
(II) Describe an identity function: $f: R \to R$

"output = input" $f(x) = x$ (domain = range)

(III) $f: R \to R^2$ not well-defined



$f = (2, 3)$



$f: R^2 \to R$

Example:

$$f(x) = \begin{cases} x+1, & x<-2 \\ -x-2; & -2 \leq x < 0 \\ x+2, & x \geq 0 \end{cases}$$

$$g(x) = \begin{cases} x^2, & x \geq 0 \\ 1, & x<0 \end{cases}$$

$f \circ g - ?$

$$f[g] = \begin{cases} \cancel{g+1, \ g<-2} & \text{– no such } g \text{ for any } x. \\ \cancel{-g-2, \ -2 \leq g < 0} & \text{– no such } g \text{ for any } x \\ g+2, \ g \geq 0 & \end{cases}$$

$x<0, \ g=1, \ f(g)=3$

$x \geq 0, \ g=x^2, \ f(g)=x^2+2$

$$f \circ g = \begin{cases} 3, & x<0 \\ x^2+2, & x \geq 0 \end{cases}$$

Ex. Show that $f(x) = \dfrac{x}{1-x}$ and $g(x) = \dfrac{x}{1+x}$ are inverses of each other.

$$f \circ g = \frac{g}{1-g} = \frac{\frac{x}{1+x}}{1-\frac{x}{1+x}} = \frac{x}{1+x-x} = x$$

$f^{-1} \circ f = x$! $\qquad g \circ f = x = f \circ g$