Name: ____________________ Mark: __________/25

Student ID: ____________________

UNIVERSITY OF OTTAWA
DEPARTMENT OF MATHEMATICS & STATISTICS

MAT 2342 (Fall 2019)
Midterm 1 (Monday, October 21, 2019)

Time: 90 minutes (no cellphones, notes, books, talking).

MARKS

(5) 1. Consider the following linear dynamical system $V_{k+1} = AV_k$ for $k \geq 0$. Find the dominant eigenvalue and approximate V_k after many years.

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 0 & 2 \\ 2 & 0 & -1 \end{bmatrix} \qquad V_0 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

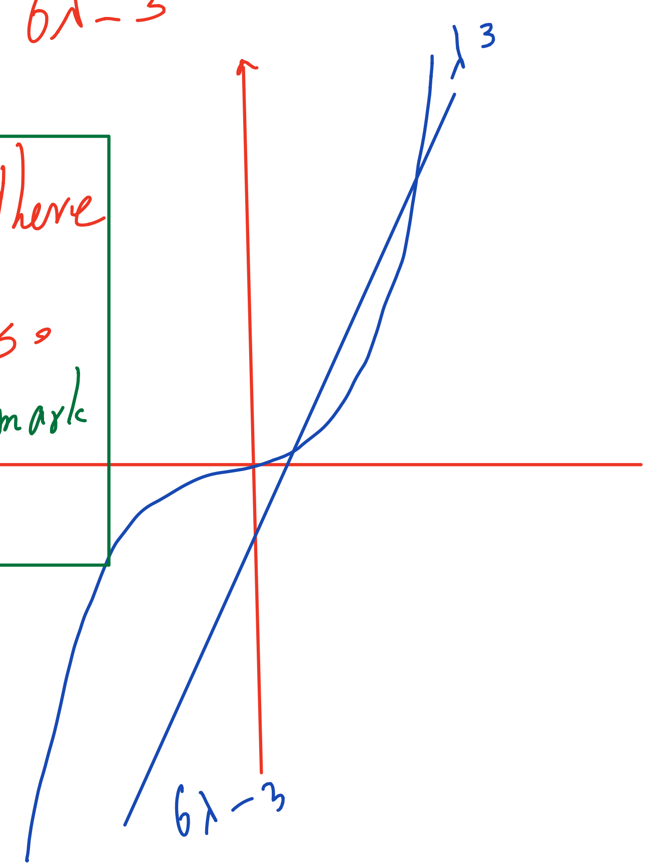
It is a bonus question and students need only to show that A is diagonalizable (3 marks)

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -1 & 2 \\ -1 & -\lambda & 2 \\ 2 & 0 & -1-\lambda \end{vmatrix} = -\lambda^3 + 6\lambda - 3 = 0$$ 2marks

$-\lambda^3 + 6\lambda - 3 = 0 \Rightarrow \lambda^3 = 6\lambda - 3$

based on the figure, there are three roots and so A is diagonalizable

1 mark



(3) 2. Let a linear dynamical system $V_{k+1} = AV_k$ be given for $k \geq 0$. Show that it is a saddle point if

$$A = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ 5 & \frac{1}{2} \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 1/4 - \lambda & 1/4 \\ 5 & 1/2 - \lambda \end{vmatrix}$$

1mark

$$= (1/4 - \lambda)(1/2 - \lambda) - 5/4 = \lambda^2 - 3/4\, \lambda - 9/8$$

$$= (\lambda - 3/2)(\lambda + 3/4)$$

1mark

$\Rightarrow \lambda_1 = 3/2$ $\quad |\lambda_1| > 1$ & $|\lambda_2| < 1$ 1mark

$\lambda_2 = -3/4$

$\Rightarrow$ saddle point.

(6) 3. Let matrix A be the transition matrix of a Markov chain. Show that A is not regular but still the steady-state-vector exists.

$$A = \begin{bmatrix} \frac{1}{3} & 0 \\ \frac{2}{3} & 1 \end{bmatrix}$$

For A to be regular, there should be an m s.th all the entries of A^m are positive. We can prove that the entry $(1,2)$ of all powers of A is zero.

for random a, b, c, d, e, f we have 3 marks

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}\begin{pmatrix} d & 0 \\ e & f \end{pmatrix} = \begin{pmatrix} ad & 0 \\ bd+ce & cf \end{pmatrix}$$

$\lambda = 1 \Rightarrow (A - I)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -2/3 & 0 \\ 2/3 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ✓

$\Rightarrow u_1 = 0 \quad u_2 = 1$ so $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is the only eigenvector corresponding to $\lambda = 1$ with sum of the entries $= 1$.

3 marks

so it is a steady-state vector

(5) 4. Consider the following linear programing P.

$$\text{maximize } -x+y \;\text{ s.th } \begin{cases} 2x-3y \le -1 \\ -x-y \le -3 \end{cases}.$$

(a) Find the feasible region of P.

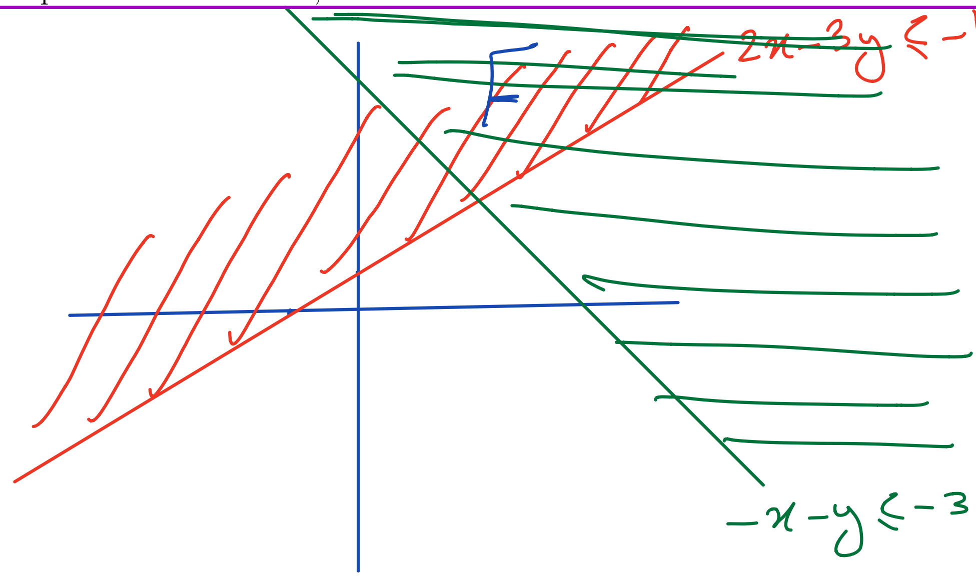
(b) Find the feasible region of the dual P^* and determine if P has optimal solutions.

(c) If the optimal solution exists, use either P or P^* to find it.

$2x-3y=-1$

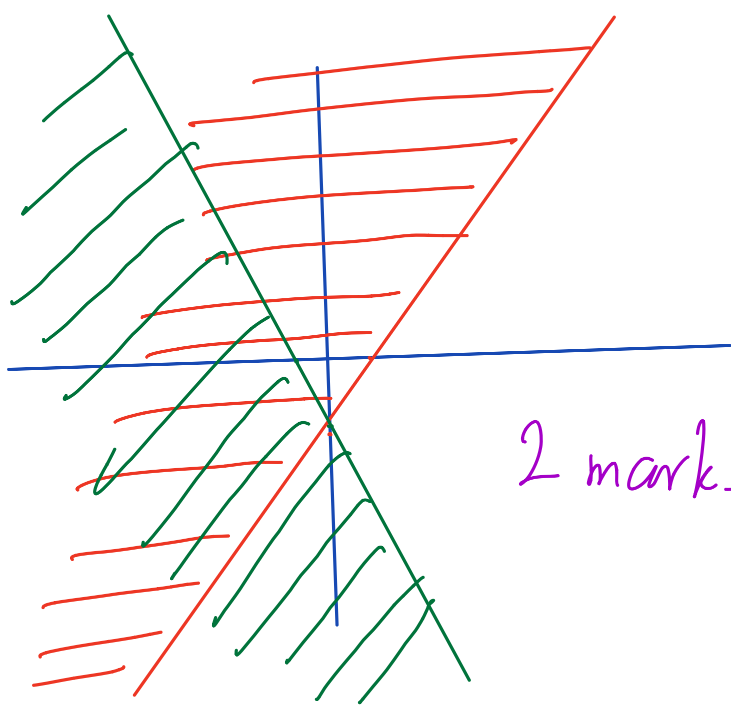
$-x-y=-3$

2 marks



dual: $2x-y \geq -1$

$-3x-y \geq 1$



2 marks

F^* is empty.

So no optimal solution for P.

1 mark

(6) 5. Consider the following linear programing.

$$\text{minimize } 2x_1 + 4x_2 + 2x_3 \text{ s.th } \begin{cases} -x_1 + x_2 + x_3 \geq 2 \\ 2x_1 + x_2 - x_3 \geq 1 \end{cases} . \quad (1)$$

Use the dual to find the optimal solution.

$x_1 - x_2 - x_3 \leq -2$

$-2x_1 - x_2 + x_3 \leq -1$

max $-2x_1 - 4x_2 - 2x_3$ (2)

3 marks

$\rightsquigarrow$

$y_1 - 2y_2 \geq -2$

$-y_1 - y_2 \geq -4$

$-y_1 + y_2 \geq -2$ (3)

min $-2y_1 - y_2$

$\iff$

$-y_1 + 2y_2 \leq 2$

$y_1 + y_2 \leq 4$

$y_1 - y_2 \leq 2$ (4)

max $2y_1 + y_2$

M	y_1	y_2	z_1	z_2	z_3	b
	-1	2	1	0	0	2
	1	1	0	1	0	4
	1	-1	0	0	1	2
1	-2	-1				0

$y_1 \leq 4$

$\boxed{y_1 \leq 2}$

M	y_1	y_2	z_1	z_2	z_3	b
$R_1 + R_3$	0	1	1	0	1	4
$\Rightarrow R_2 - R_3$	0	2	0	1	-1	2
R_3	1	-1	0	0	1	2
$R_4 + 2R_3$	0	-3	0	0	2	4

$y_2 \leq 4$

$\boxed{2y_2 \leq 2}$

M	y_1	y_2	z_1	z_2	z_3	b
$R_1 - R_2/2$	0	0	1	$-1/2$	$3/2$	3
$R_2/2$	0	1	0	$1/2$	$-1/2$	1
$R_3 + R_2/2$	1	0	0	$1/2$	$1/2$	3
$R_4 + \frac{3}{2}R_2$	0	0	0	$3/2$	$1/2$	7

3 marks

7 max of (4)

$\Rightarrow$ -7 min of (3)

$\Rightarrow$ -7 max of (2)

$\Rightarrow$ 7 min of (1)

This part is a must do. But that is fine, no mark for that.