Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

MAT 2122, Fall 2019 – Midterm exam 2 (practice) **Instructor: A. Tikuisis**

Solutions

Read the following instructions:

- The use of cellphones, electronic devices (including calculators), and course notes is strictly forbidden. All phones and electronic devices must be turned off and kept in your bags: do not leave them on you. If you are seen to have an electronic device on your person, we may ask you to leave the exam immediately, and fraud allegations could be made, which could lead to a mark of 0 (zero) on this midterm.
- The duration of this midterm is 75 minutes.
- This is a closed book midterm containing **5 questions**.
- There is an additional blank page at the end of this exam that you may use as scrap paper. If you run out of space, you may use this page or the backs of pages. Clearly indicate where to find your answer.
- Do not detach the pages of this test, apart from the last (blank) page. If you detach the last page, do not use it for your submitted answers.
- You must give clear and complete solutions, with calculations, explanations and justifications. Make sure that your answer is clearly indicated; you must convince me that you understand your solution in order to receive full marks.

By signing below, you acknowledge that you are required to respect the above statements.

Signature: ____________________________

THIS SPACE IS RESERVED FOR THE MARKER:

Question	1	2	3	4	Total
Mark					
Out of	9	15	15	11	50

1. Multiple choice. Write your answer clearly in the blank below the question, or write "X" to indicate blank. Each question is worth **3 marks** and has exactly one correct answer. A correct solution is worth 3 marks, an incorrect or blank solution is worth 0 marks, and "X" (intentional blank) is worth 1 mark.

(i) Let $R := [a,b] \times [c,d]$ and $f : R \to \mathbb{R}$ be a function. Under which of the following conditions is $\int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy$?

(A) If f is a bounded C^1 function.

(B) (A) and more generally, if f is a bounded continuous function.

(C) (B) and more generally, if f is a bounded function and the set of points where f is not continuous is contained in a finite union of graphs of continuous functions, and $\int_a^b f(x,y)\, dx$ exists for each $y \in [c,d]$ and $\int_c^d f(x,y)\, dy$ exists for each $x \in [a,b]$.

(D) (C) and more generally, if f is an integrable function.

(E) (D) and more generally, if f is any bounded function.

Solution: (C)

(ii) Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2, f : \mathbb{R}^2 \to \mathbb{R}$ be C^1 functions and let $R, S \subseteq \mathbb{R}^2$ be elementary regions. Suppose that ϕ is injective on R and $\phi(R) = S$, and that

$$\int\int_S f(x,y)\, dx\, dy = 6, \quad \int\int_S f(\phi(x,y))|\det D\phi(x,y)|\, dx\, dy = -12,$$
$$\int\int_R 1\, dx\, dy = 3, \quad \int\int_R |\det D\phi(x,y)|\, dx\, dy = 2.$$

What is the average of f on S?

(A) 2.

(B) 3.

(C) -4.

(D) 6.

(E) -6.

Solution: (B)

Using change-of-variables, we have

$$\text{vol}(S) = \int\int_S 1\,dx\,dy = \int_R |\det D\phi(x,y)|\,dx\,dy = 2,$$

and so the average of f on S is

$$\frac{1}{\text{vol}S}\int\int_S f(x,y)\,dx\,dy = \frac{6}{2} = 3.$$

(iii) Which of the following is equal to $\int_0^1 \int_{\sqrt{x}}^1 f(x,y)\,dy\,dx$?

(A) $\int_0^1 \int_{y^2}^1 f(x,y)\,dy\,dx$.

(B) $\int_0^1 \int_0^{y^2} f(x,y)\,dy\,dx$.

(C) $\int_0^1 \int_{y^2}^1 f(x,y)\,dx\,dy$.

(D) $\int_0^1 \int_0^{y^2} f(x,y)\,dx\,dy$.

(E) None of the above.

Solution: (D)

The region of integration is

$$R := \{(x,y) : x \in [0,1],\ y \in [\sqrt{x}, 1]\}.$$

In other words, it is given by the constraints

$$0 \le x \le 1, \quad \sqrt{x} \le y \le 1.$$

The constraint $\sqrt{x} \le y$ can be rewritten $x \le y^2$, and we note that if $y \le 1$ and $x \le y^2$ then automatically $x \le 1$. Hence these can be rewritten

$$0 \le y \le 1, \quad 0 \le x \le y^2,$$

and so the integral can be rearranged as

$$\int_0^1 \int_0^{y^2} f(x,y)\,dx\,dy.$$

2. Let $T \subseteq \mathbb{R}^2$ be the triangle with vertices $(0,0)$, $(0,1)$, $(1,0)$.

(i) Give formulas for path(s) which cover the boundary of T. 4

Solution: The line from $(0,0)$ to $(0,1)$ is given by $c_1(t) := (0,t)$ for $t \in [0,1]$.
The line from $(0,0)$ to $(1,0)$ is given by $c_2(t) := (t,0)$ for $t \in [0,1]$.
The line from $(0,1)$ to $(1,0)$ is given by $c_3(t) := (t, 1-t)$ for $t \in [0,1]$.

(ii) Determine all points in T where the function $f(x,y) := 3x^2 - xy + y - 2x$ attains its minimum. 11

Solution: Begin with the first derivative test to find critical points in the interior of T. We have $\nabla f = (6x - y - 2, -x + 1)$, so if $\nabla f = (0,0)$ then

$$\begin{aligned} -x + 1 = 0 \;\Rightarrow\; & \qquad x = 1 \\ 6x - y - 2 = 0 \;\Rightarrow\; & \qquad 6 - y - 2 = 0 \;\Rightarrow\; y = 4. \end{aligned}$$

This gives the point $(1,4)$ which is outside of the triangle T.

Now we look for potential minima on the boundary, using the parametrization from (i). We have $(f \circ c_1)(t) = t$ $(t \in [0,1])$ and by inspection this attains its minimum, 0, at $t = 0$. We have $(f \circ c_2)(t) = 3t^2 - 2t$, so

$$(f \circ c_2)'(t) = 6t - 2,$$

and setting this equal to 0 gives $t = \frac{1}{3}$. Here we have

$$(f \circ c_2)(\tfrac{1}{3}) = 3(\tfrac{1}{3})^2 - 2(\tfrac{1}{3}) = -\tfrac{1}{3}.$$

Next, we have

$$(f \circ c_3)(t) = 3t^2 - t(1-t) + (1-t) - 2t = 4t^2 - 4t + 1.$$

so that

$$(f \circ c_3)'(t) = 8t - 4,$$

and setting this equal to 0 gives $t = \frac{1}{2}$. Here we have

$$(f \circ c_3)(\tfrac{1}{2}) = 4(\tfrac{1}{2})^2 - 4(\tfrac{1}{2}) + 1 = 1 - 2 + 1 = 0.$$

Finally, we check the corners of the triangle:

$$f(0,0) = 0, \quad f(0,1) = 1, \quad f(1,0) = 1.$$

Of all these points, the minimum is $-\frac{1}{3}$ occurring at $c_2(\frac{1}{3}) = (\frac{1}{3}, 0)$.

3. Use the Lagrange multiplier method to show that the maximum of

$$f(x,y,z) := x + 3z \quad \text{on} \quad S := \{(x,y,z) : x^2 + y^2 = 1 \text{ and } x + z^3 = 1\}$$

occurs at the points $(0,1,1)$ and $(0,-1,1)$, and at no other points. (*You may use* $3\sqrt[3]{2} = 3.779\ldots$.) 15

Solution: We write $S = \{(x,y,z) : g_1(x,y,z) = 1,\ g_2(x,y,z) = 1\}$ where

$$g_1(x,y,z) = x^2 + y^2 \quad \text{and} \quad g_2(x,y,z) = x + z^3.$$

We compute

$$\nabla f = (1,0,3), \nabla g_1 = (2x, 2y, 0), \nabla g_2 = (1, 0, 3z^2).$$

The Lagrange multiplier theorem tells us to find critical points by solving

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2, \quad g_1 = 1, \quad g_2 = 1$$

for $\lambda_1, \lambda_2 \in \mathbb{R}$. This gives the equations

$$1 = 2\lambda_1 x + \lambda_2 \tag{1}$$
$$0 = 2\lambda_1 y \tag{2}$$
$$3 = 3\lambda_2 z^2 \tag{3}$$
$$x^2 + y^2 = 1 \tag{4}$$
$$x + z^3 = 1 \tag{5}$$

From (2) we get $\lambda_1 = 0$ or $y = 0$.

If $\lambda_1 = 0$ then by (1), $\lambda_2 = 1$ and then (3) implies $z = \pm 1$. If $z = -1$ then (5) implies $x = 2$, which is not possible by (4). If $z = 1$ then (5) implies $x = 0$ and then by (4), $y = \pm 1$.

This gives the points $(0, \pm 1, 1)$, and at these points we have $f = 3$.

If $y = 0$ then by (4), $x = \pm 1$. If $x = 1$ then by (5), $z = 0$, so we get the point $(1,0,0)$ and at this point, $f = 1$. If $x = -1$ then by (5), $z = \sqrt[3]{2}$, so we get the point $(-1, 0, \sqrt[3]{2})$ and at this point, $f = -1 + 3\sqrt[3]{2} = 2.779...$

Hence the maximum is 3 and it occurs at $(0, \pm 1, 1)$.

4. Let $B := \{(x, y, z) : x^2 + y^2 + z^2 \leq 1, z \geq 0\}$.

(i) Write down a set $A \subseteq \mathbb{R}^3$ such that

$$B = \{(r\sin(\phi)\cos(\theta), r\sin(\phi)\sin(\theta), r\cos(\phi)) : (r, \theta, \phi) \in A\}.$$ 4

Solution: $r^2 = x^2+y^2+z^2 \leq 1$, so we ask that $r \in [0, 1]$. Also, $r\cos(\phi) \geq 0$, so we ask that $\phi \in [0, \frac{\pi}{2}]$. Thus,

$$A = [0, 1] \times [0, 2\pi] \times [0, \frac{\pi}{2}].$$

(ii) Use spherical coordinates to evaluate

$$\int\int\int_B z^2\, dx\, dy\, dz.$$

(*You may use that* $\frac{d}{dt}(\cos(t))^3 = -3(\cos(t))^2 \sin(t)$.) 7

Solution: Using change-of-variables for spherical coordinates,

$$\begin{aligned}
\int\int\int_B z^2\, dx\, dy\, dz &= \int_0^{2\pi}\int_0^1\int_0^{\pi/2} (r\cos(\phi))^2 r^2 \sin(\phi)\, d\phi\, dr\, d\theta \\
&= 2\pi \int_0^1 r^4\, dr \int_0^{\pi/2} (\cos\phi)^2 \sin\phi\, d\phi \\
&= 2\pi\, \frac{r^5}{5}\Big|_0^1 \left(-\frac{1}{3}(\cos\phi)^3 \Big|_0^{\pi/2} \right) \\
&= -2\pi \cdot \frac{1}{5} \cdot \frac{1}{3}(-1) = \frac{2\pi}{15}.
\end{aligned}$$

(the last page)