Solutions to Math 104 Practice Exam

1. Suppose that $f(x)$ is a function such that $f(3)=1$ and $f'(3)=-2$. Let $g(x)=\ln\left(x^2+\sqrt{f(x)}\right)$. Find $g'(3)$. [10 Marks]

Solution: $g'(x)=\dfrac{1}{x^2+[f(x)]^{\frac{1}{2}}}\cdot\left\{2x+\tfrac{1}{2}[f(x)]^{-\frac{1}{2}}\cdot f'(x)\right\} = \dfrac{1}{x^2+\sqrt{f(x)}}\cdot\left\{2x+\dfrac{f'(x)}{2\sqrt{f(x)}}\right\}$

$$g'(3)=\frac{1}{3^2+\sqrt{f(3)}}\cdot\left\{2\cdot 3+\frac{f'(3)}{2\sqrt{f(3)}}\right\} = \frac{1}{9+\sqrt{1}}\cdot\left\{6+\frac{-2}{2\sqrt{1}}\right\} = \frac{1}{9+1}\cdot\left\{6-\tfrac{2}{2}\right\} = \tfrac{1}{10}\cdot[6-1] = \tfrac{1}{2}$$

2. Find an equation for the tangent line to the curve $(x^3+y^3)^2=4xy^2$ at the point $(1,1)$. [10 Marks]

Solution: $\dfrac{d}{dx}(x^3+y^3)^2=\dfrac{d}{dx}(4xy^2)$

$$2(x^3+y^3)\cdot\frac{d}{dx}(x^3+y^3) = 4\left(x\frac{d}{dx}(y^2)+y^2\frac{d}{dx}(x)\right)$$

$$2(x^3+y^3)\cdot\left(3x^2+3y^2\frac{dy}{dx}\right) = 4\left(x\cdot 2y\frac{dy}{dx}+y^2\cdot 1\right)$$

$$2(x^3+y^3)\cdot 3\left(x^2+y^2\frac{dy}{dx}\right) = 4\left(2xy\frac{dy}{dx}+y^2\right)$$

Plugging in $x=1$, $y=1$,

$$2\cdot 2\cdot 3\left(1+\frac{dy}{dx}\right) = 4\left(2\frac{dy}{dx}+1\right) \;\Rightarrow\; 12+12\frac{dy}{dx} = 8\frac{dy}{dx}+4 \;\Rightarrow\; 4\frac{dy}{dx}=-8.$$

The slope of the tangent line is $m = \left.\dfrac{dy}{dx}\right|_{(1,1)} = -2.$

The equation of the tangent line is therefore $y-1=-2(x-1)$ or $y=-2x+3$.

3. Let $f(x)=x\ln x$. Compute the 4[th] degree Taylor polynomial of $f(x)$ about $a=5$. [10 Marks]

Solution:

$f(x)=x\ln x,\qquad f(5)=5\ln 5;$

$f'(x)=x\cdot\dfrac{1}{x}+\ln x\cdot 1=1+\ln x,\qquad f'(5)=1+\ln 5;$

$f''(x)=0+\dfrac{1}{x}=x^{-1},\qquad f''(5)=\dfrac{1}{5};$

$f'''(x)=-x^{-2}=-\dfrac{1}{x^2},\qquad f'''(5)=-\dfrac{1}{25};$

$f''''(x)=2x^{-3}=\dfrac{2}{x^3},\qquad f''''(5)=\dfrac{2}{125}.$

The 4[th] degree Taylor polynomial of $f(x)$ about $a=5$ is therefore

$$\begin{aligned}T_4(x) &= f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\frac{f''''(a)}{4!}(x-a)^4\\ &= f(5)+\frac{f'(5)}{1!}(x-5)+\frac{f''(5)}{2!}(x-5)^2+\frac{f'''(5)}{3!}(x-5)^3+\frac{f''''(5)}{4!}(x-5)^4\\ &= 5\ln 5+\frac{1+\ln 5}{1}(x-5)+\frac{\frac{1}{5}}{2}(x-5)^2+\frac{-\frac{1}{25}}{6}(x-5)^3+\frac{\frac{2}{125}}{24}(x-5)^4\\ &= 5\ln 5+(1+\ln 5)(x-5)+\frac{1}{10}(x-5)^2-\frac{1}{150}(x-5)^3+\frac{1}{1500}(x-5)^4.\end{aligned}$$

4. The demand q for a product is a function of the unit price p. The *elasticity of demand* $E(p)$ is defined by

$$E(p) = -\frac{p}{q}\frac{dq}{dp}.$$

Assume that p and q satisfy the demand equation $q = 4\sqrt{\frac{60-p}{p}}$ for $0 < p < 60$.

(a) Find the elasticity of demand when $p = \$50$. [5 Marks]

Solution: $$\frac{dq}{dp} = 4\frac{d}{dp}\left(\frac{60-p}{p}\right)^{\frac{1}{2}} = 4\cdot\frac{1}{2}\left(\frac{60-p}{p}\right)^{-\frac{1}{2}}\frac{d}{dp}\left(\frac{60-p}{p}\right) = 2\left(\frac{60-p}{p}\right)^{-\frac{1}{2}}\cdot\frac{p\cdot\frac{d}{dp}(60-p)-(60-p)\cdot\frac{d}{dp}(p)}{p^2}$$

$$= 2\left(\frac{60-p}{p}\right)^{-\frac{1}{2}}\cdot\frac{p\cdot(-1)-(60-p)\cdot 1}{p^2} = 2\left(\frac{60-p}{p}\right)^{-\frac{1}{2}}\cdot\frac{-p-60+p}{p^2} = 2\cdot\frac{-60}{p^2\sqrt{\frac{60-p}{p}}}.$$

Since $\sqrt{\frac{60-p}{p}} = \frac{1}{4}p$, we have $\frac{dq}{dp} = \frac{-120}{p^2\sqrt{\frac{60-p}{p}}} = -\frac{120}{p^2\cdot\left(\frac{1}{4}q\right)} = -\frac{480}{p^2 q}$, so the elasticity of demand is

$$E = -\frac{p}{q}\frac{dq}{dp} = -\frac{p}{q}\left(-\frac{480}{p^2 q}\right) = \frac{480}{pq^2} = \frac{480}{p\cdot 16\left(\frac{60-p}{p}\right)} = \frac{30}{60-p}.$$

When $p = \$50$, the elasticity of demand is $E = \frac{30}{60-50} = 3$.

(b) Use the result of part (a) to estimate the percentage change of demand when the price increases 0.7% from \$50. [5 Marks]

Solution: Since the elasticity of demand is $E = 3$ at the price level of $p = \$50$, this means that every 1% increase in price above the \$50 level will result in an approximate 3% decrease in demand. So if the price increases by 0.7% from \$50, the demand will go down by about $3\times 0.7\% = 2.1\%$. That is the change in demand will be approximately -2.1%.

5. Let $f(x) = x^2 - 5x + 2 - 3\ln x$. Find all values of x at which $f(x)$ attains a local maximum, and all values of x at which $f(x)$ attains a local minimum. Justify in each case that you have found a local maximum or local minimum. [10 Marks]

Solution: At a critical point, $f'(x) = 0$, so $2x - 5 - \frac{3}{x} = 0$. Therefore $2x^2 - 5x - 3 = 0$, or $(2x+1)(x-3) = 0$.

So either $2x+1 = 0$ or $x - 3 = 0$. The solutions are therefore $x = -\frac{1}{2}$ and $x = 3$. The negative solution must be rejected since $\ln x$ is undefined for $x \le 0$, so the only critical value is $x = 3$.

Since $f''(x) = 2 + \frac{3}{x^2} > 0$, the curve is always concave up and so, by the Second Derivative Test, the critical value $x = 3$ gives rise to a local minimum. Since there are no other critical values, the function $f(x)$ never attains a local maximum.

6. Let $f(x) = x^4 e^{-x}$. Determine where the graph of $f(x)$ is concave up. [10 Marks]

Solution: $f'(x) = x^4\frac{d}{dx}(e^{-x}) + e^{-x}\frac{d}{dx}(x^4) = x^4\cdot(-e^{-x}) + e^{-x}\cdot(4x^3) = e^{-x}(4x^3 - x^4)$

$$f''(x) = e^{-x}\frac{d}{dx}(4x^3 - x^4) + (4x^3 - x^4)\frac{d}{dx}(e^{-x}) = e^{-x}\cdot(12x^2 - 4x^3) + (4x^3 - x^4)\cdot(-e^{-x})$$

$$= e^{-x}\cdot[(12x^2 - 4x^3) - (4x^3 - x^4)] = e^{-x}\cdot[12x^2 - 8x^3 + x^4] = x^2 e^{-x}(12 - 8x + x^2) = x^2 e^{-x}(x-2)(x-6).$$

The graph of $f(x)$ is concave up when its second derivative is positive, i.e. when $f''(x) > 0$.

Since $f''(x) = x^2 e^{-x}(x-2)(x-6)$ is the product of four factors, of which the first two (x^2 and e^{-x}) are never negative, the sign of $f''(x)$ is completely determined by the signs of the last two factors, i.e. $(x-2)$ and $(x-6)$. These two factors are equal to zero at $x = 2$ and $x = 6$ respectively, so we have three cases to consider;

		$x-2$	$x-6$	$f''(x) = x^2 e^{-x}(x-2)(x-6)$
Case 1:	$x < 2$	$-$	$-$	$+$
Case 2:	$2 < x < 6$	$+$	$-$	$-$
Case 3:	$x > 6$	$+$	$+$	$+$

We see that $f''(x) > 0$ only when $x < 2$ or $x > 6$.

So the graph of $f(x)$ is concave up only on the intervals $(-\infty, 2)$ and $(6, \infty)$.

7. A company sells q desks per month. The monthly cost is $C(q) = 600 + 30q$ and the demand equation is $p = 180 - 30\ln q$ where q is the number of desks sold per month at price p. Find the maximum profit, the production level that will realize the maximum profit, and the price the company should charge for each desk. [10 Marks]

Solution: The profit is given by $P = R - C$ (revenue minus cost), where the revenue is $R = pq = (180 - 30\ln q)q$.

$$P(q) = R - C = (180 - 30\ln q)q - (600 + 30q) = (180q - 30q\ln q) - 600 - 30q = 150q - 30q\ln q - 600.$$

$$\frac{dP}{dq} = 150 - 30(q \cdot \frac{1}{q} + \ln q \cdot 1) - 0 = 150 - 30(1 + \ln q) = 150 - 30 - 30\ln q = 120 - 30\ln q.$$

At a critical point, $\frac{dP}{dq} = 0$, so $120 = 30\ln q$ or $\ln q = 4$, so $q = e^4$.

Since $\frac{d^2R}{dq^2} = 0 - \frac{60}{q} = -\frac{60}{e^4} < 0$, the curve is concave down, so the critical point gives a local maximum (by the Second Derivative Test). The maximum profit is then

$$P(e^4) = 150e^4 - 30e^4\ln e^4 - 600 = 150e^4 - 30e^4 \cdot 4 - 600 = 150e^4 - 120e^4 - 600 = 30e^4 - 600.$$

The corresponding price is $p = 180 - 30\ln q = 180 - 30\ln e^4 = 180 - 30 \cdot 4 = 180 - 120 = 60.$

So the maximum profit is $P = 30e^4 - 600$, the production level is $q = e^4$, and the price is $p = \$60$.

8. (a) Use a linear approximation to obtain an estimate of $\sqrt{26}$. [5 Marks]

Solution: Let $f(x) = \sqrt{x}$. We want to approximate $f(26) = \sqrt{26}$.

Since $f(x) = \sqrt{x} = x^{\frac{1}{2}}$, therefore $f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$. Let $a = 25$.

So $f(a) = f(25) = \sqrt{25} = 5$, and $f'(a) = f'(25) = \frac{1}{2\sqrt{25}} = \frac{1}{10}$.

The linear approximation to $f(x)$ near $a = 25$ is

$$f(x) \approx f(a) + f'(a)(x-a) = f(25) + f'(25)(x-25) = 5 + \frac{1}{10}(x-25).$$

Therefore $\sqrt{26} = f(26) \approx 5 + \frac{1}{10}(26-25) = 5 + \frac{1}{10} = 5.1.$

(b) Use a quadratic approximation to obtain a better estimate. [5 Marks]

Solution: Find the second derivative $f''(x) = -\frac{1}{4}x^{-\frac{3}{2}} = -\frac{1}{4\sqrt{x}^3}$.

So $f''(a) = f''(25) = -\frac{1}{4\sqrt{25}^3} = -\frac{1}{4 \cdot 5^3} = -\frac{1}{500}$.

The quadratic approximation to $f(x)$ near $a = 25$ is

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 = f(25) + f'(25)(x-25) + \frac{1}{2}f''(25)(x-25)^2$$

$$= 5 + \frac{1}{10}(x-25) - \frac{1}{2} \cdot \frac{1}{500}(x-25)^2 = 5 + \frac{1}{10}(x-25) - \frac{1}{1000}(x-25)^2.$$

Therefore $\sqrt{26} = f(26) \approx 5 + \frac{1}{10}(26-25) - \frac{1}{1000}(26-25)^2 = 5 + \frac{1}{10} - \frac{1}{1000} = 5.000 + 0.100 - 0.001 = 5.099.$

9. A zero-coupon bond is a bond that is sold for less than its face value and has no periodic interest payments. Instead, the bond is redeemed for its face value at maturity. Thus, in a sense, interest is paid at maturity. Suppose that a zero-coupon bond sells for \$400 and can be redeemed in 16 years for its face value of \$1,000.

(a) What is the nominal interest rate of the bond under continuous compounding? (A calculator-ready answer is enough.) [5 Marks]

Solution: The amount A that the principal P will grow to in t years under continuous compounding at a nominal interest rate of $100r$ % is given by

$$A = Pe^{rt}.$$

Here we have $A = 1000$, $P = 400$, $t = 16$, and we need to solve for r.

$$1000 = 400\,e^{r\cdot 16}$$

$$\frac{1000}{400} = e^{16r}$$

$$\ln\frac{10}{4} = 16r$$

$$r = \frac{1}{16}\ln 2.5 \approx 0.0573$$

The nominal interest rate is $100r\,\% = \frac{100}{16}\ln 2.5 \approx 5.73\%$ per year.

(b) What is the doubling time under the interest rate scheme in part (a)? (Again, a calculator-ready answer is enough.) [5 Marks]

Solution: If the doubling time is t years, then the amount to which the principal P will grow will be $A = 2P$. Therefore

$$2P = Pe^{rt}$$

$$2 = e^{rt}$$

$$\ln 2 = rt$$

$$t = \frac{\ln 2}{r}$$

Using the value of r from part (a) gives a doubling time of

$$t = \frac{\ln 2}{\frac{1}{16}\ln 2.5} = \frac{16\ln 2}{\ln 2.5} \approx 12.1 \text{ years.}$$

10. It is desired to use Newton's Method to find the value of x where the function $f(x) = x^4 - x^3 - 2x + 1$ has its absolute maximum. Find x_1 if $x_0 = 1$. Note that you are only required to find x_1. [10 Marks]

Solution: Since $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty}(x^4 - x^3 - 2x + 1) = \lim_{x\to\pm\infty} x^4(1 - \frac{1}{x} - \frac{2}{x^3} + \frac{1}{x^4}) = \infty$, there can be no absolute maximum. There will only be an absolute minimum and it must occur at a critical point, where $f'(x) = 0$. Since $f'(x) = 4x^3 - 3x^2 - 2$, we must solve the equation $4x^3 - 3x^2 - 2 = 0$.

Let $g(x) = 4x^3 - 3x^2 - 2$. Therefore, we need to approximate the zeros of $g(x)$ by Newton's Method.

Since $g'(x) = 12x^2 - 6x$, we have

$$x_1 = x_0 - \frac{g(x_0)}{g'(x_0)} = 1 - \frac{g(1)}{g'(1)} = 1 - \frac{4-3-2}{12-6} = 1 - \frac{-1}{6} = \frac{7}{6} \approx 1.1667.$$