York University

Department of Electrical Engineering and Computer Science
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MATH1090B. Mid Term Test, October 31, 2019 —SOLUTIONS

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Question 1. (4 MARKS) *Syntactic proofs will NOT be accepted in this question*!

Through truth tables or related short cuts show that

$$\models_{\text{taut}} (A \equiv B) \equiv A \wedge B \vee \neg A \wedge \neg B$$

Proof. Shortcut of truth table.

Case 1) Let v be a state that causes A and B to have the *same* truth value.
Then the lhs is $\mathbf{t}$.
So is the rhs, since two $\mathbf{f}$'s cause $\neg A \wedge \neg B$ to be $\mathbf{t}$, while two $\mathbf{t}$'s cause $A \wedge B$ to be $\mathbf{t}$.

Case 2) Let v be a state that causes A and B to have *different* truth values.
Then the lhs is $\mathbf{f}$.
So is the rhs: Why? Well, ONE of A and B will be $\mathbf{f}$ and thus ONE of $\neg A$ and $\neg B$ will also be $\mathbf{f}$.
Hence both of $A \wedge B$ and $\neg A \wedge \neg B$ will be $\mathbf{f}$, and hence the entire rhs.

□

Question 2. (3 MARKS) *Syntactic proofs will NOT be accepted in this question*!

Through truth tables or related short cuts show that, for any formulas A and B, we have

$$A \wedge (\bot \equiv A) \models_{\text{taut}} B$$

Proof. By short cuts.

Let v be a state that makes the lhs $\mathbf{t}$. This entails

- A is $\mathbf{t}$

AND

- $\bot \equiv A$ is $\mathbf{t}$. But then A is $\mathbf{f}$.

Therefore there is NO v that makes the lhs $\mathbf{t}$, which is great! We do not have to do anything and the tautological implication is valid.

Recall that we *work* on a tautological implication ONLY for those v that make the entire lhs $\mathbf{t}$.

□

Question 3. (6 MARKS) Prove that the following are ***not*** wff. **The proof in each case must be by analysing either formula constructions, or the recursive definition of formulas.**

(a) $(\top\neg\bot)$

Proof. A formula is one of:

- Atomic
- Of the form $(\neg B)$
- Of the form $(B \circ C)$

Thus, whenever we have the glue "$\neg$" occur in a wff, then we have a "(" immediately to its left.

Not so in the given string, since $\neg$ has no left bracket to its left. So not a wff.

We can say the same thing via **formula constructions**: *Every time we write down* $\neg$ *during a formula construction we also write a "(" immediately to its left. So "$(\top\neg\bot)$" cannot possibly be written during such a construction, so it is not a wff.*

□

(b) ($\mathbf{p}$)

Proof. Again looking at the recursive definition above we note that we place brackets ONLY if there is glue added. (Same observation if we analyse a formula construction). So it is not possible to add brackets but no glue. (p) is not a wff. □

Question 4. (5 MARKS) Give an **Equational** proof of $\bot \vdash A$, for any formula A.

Limitations to allowed tools: May ***NOT*** use:

- Post's Theorem
- Deduction Theorem
- Resolution
- Cut Rule
- Hilbert style proof

Use of any of the listed above tools will entail that 0 MARKS will be earned in this problem.

Proof. Start with what we want to prove.

$$
\begin{aligned}
& A \\
\Leftrightarrow\ & \langle \text{thm (class)} \rangle \\
& A \vee \bot \\
\Leftrightarrow\ & \langle \text{Red. } \top \text{ MetaThm } (\bot \text{ is hyp}) + \text{Leib; Denom: } A \vee \mathbf{p} \rangle \\
& A \vee \top \qquad\qquad \text{bingo!}
\end{aligned}
$$

□

Question 5. (5 MARKS) **True** or **False**, and *WHY*?

"If $\vdash A \vee B$, then one of the following holds: $\vdash A$ or $\vdash B$"

If you claim that this is *false*, then you **MUST** argue in terms of **specific** A and B!

Answer. *False.*

Take A to be p (non bold; actual var.) and B to be $\neg p$.

Thus, we have $\vdash A \vee B$ (axiom $p \vee \neg p$), but we have neither of $\vdash p$ and $\vdash \neg p$ since if we did, then soundness would require $\models_{taut} p$ and $\models_{taut} \neg p$. □

Question 6. (5 MARKS) *Prove* via an **Equational proof** the following:

You may **NOT** use Post's Theorem!

$$\vdash ((A \wedge B) \wedge C) \equiv (A \wedge (B \wedge C)) \qquad (1)$$

You may NOT invoke "generalised associativity" for $\wedge$ as it was *never proved nor used in class*! It is not an axiom either.

Proof.

$$
\begin{aligned}
& ((A \wedge B) \wedge C) \\
\Leftrightarrow\ & \langle \text{dM} \rangle \\
& \neg(\neg(A \wedge B) \vee \neg C) \\
\Leftrightarrow\ & \langle \text{dM + Leib: Denom: } \neg(\mathbf{p} \vee \neg C) \rangle \\
& \neg((\neg A \vee \neg B) \vee \neg C) \\
\Leftrightarrow\ & \langle \text{axiom 1 + Leib: Denom: } \neg\mathbf{p} \rangle \\
& \neg(\neg A \vee (\neg B \vee \neg C)) \\
\Leftrightarrow\ & \langle \text{dM + Leib: Denom: } \neg(\neg A \vee \mathbf{p}) \rangle \\
& \neg(\neg A \vee \neg(B \wedge C)) \\
\Leftrightarrow\ & \langle \text{dM} \rangle \\
& (A \wedge (B \wedge C))
\end{aligned}
$$

□

Question 7. (5 MARKS) Use a **Hilbert style** proof, but **NOT** Post's Theorem, to prove the following for any choice of formulas A, B and C:

$$\vdash (A \to B) \to (B \to C) \to (A \to C)$$

Limitation to Tools: You may **NOT** use an Equational proof.

Proof. By DThm, it suffices to prove

$$A \to B \vdash (B \to C) \to (A \to C)$$

By DThm again, it suffices to prove

$$A \to B, B \to C \vdash A \to C$$

By DThm yet again, it suffices to prove

$$A \to B, B \to C, A \vdash C$$

Let's prove this (we are required to use a Hilbert-style proof).

(1)	$A \to B$	$\langle \text{hyp} \rangle$
(2)	$B \to C$	$\langle \text{hyp} \rangle$
(3)	A	$\langle \text{hyp} \rangle$
(4)	B	$\langle (1, 3) + \text{MP} \rangle$
(5)	C	$\langle (2, 4) + \text{MP} \rangle$

□