UNIVERSITY OF TORONTO

Faculty of Arts and Science

December 2018 EXAMINATIONS

STA347H1F L0101

Duration - 2 hours

Examination Aids: A non-programmable calculator

Last name: ______________________________

First name: ______________ **Middle name:** ______________

Student number: ______________________________

Instruction

1. If a question asks you to do some calculations or derivations, you must show your work to receive full credit. Simplify expressions as much as you can.
2. If you do not have enough space, use another booklet and refer it correctly.
3. Some problems could be wrong. If it happens, correct questions by yourself after arguing why the problem is wrong.
4. Please do not pull the pages apart unless necessary. If you do so, print your name and sign all the pages.
5. You can use either pen or pencil for the test. **But please be aware that you are not allowed to dispute any credit after the test is returned if you use pencil.**
6. You are allowed to use any theorem covered in class as well as any statement in other questions in this exam. For example, in question x (y), you can use the result of question x (z) without a proof.
7. Some useful **information** are on page 3. You can use pages 4 and 5 as scratch space.
8. There are 6 problems on 5 pages including scratch space, and the total is 100 marks. Please hand in your exam paper and booklets when the time is up.

Problem 1 (8 marks)**.** Sequences of events A_n, B_n, C_n satisfy $P(A_n) \to 1$, $P(B_n) \to 1/2$, $P(C_n) \to 1/3$ as $n \to \infty$ and B_n and C_n are independent for each n. Compute $\limsup_{n\to\infty} P(A_n \cap B_n \cap C_n)$.

Problem 2 (8 marks)**.** Four events A, B, C, D having positive probabilities satisfy $C \cap D = \emptyset$, $P(A \mid C) \geq P(B \mid C)$ and $P(A \mid D) \geq P(B \mid D)$. Prove or disprove $P(A \mid C \cup D) \geq P(B \mid C \cup D)$.

Problem 3. Let $Z_1, \ldots, Z_4$ be i.i.d. $N(0,1)$. Let $Y = Z_1 Z_2 + Z_3 Z_4$.
(a) [8 marks] Prove that the moment generating function of Y is $1/(1-t^2)$ for $|t| < 1$.
(b) [8 marks] Compute the mean, variance, skewness and kurtosis of Y.
(c) [8 marks] Approximate $P(Y > 7)$ using inequalities as accurate as possible.

Problem 4. The number of clients visiting an express coffee take-out shop for a certain time period follows a Poisson distribution with parameter 10. Each client independently orders either cafe latte or house blend at this time period. In this time period, the probability that a client is female is 60% and probability of male is 40%. Each female client orders a cafe latte with probability 2/3 or a house blend with probability 1/3. While each male client orders a cafe latte or house blend with equal probability.
(a) [8 marks] Compute the probability of more than two cafe lattes were sold within this time period.
(b) [8 marks] A client orders a house blend. Compute the probability that the client is female.

Problem 5. Two random variables X and Y have joint density

$$\mathrm{pdf}_{X,Y}(x, y) = c \cdot xy(1 - x^2 - y^2)1(x > 0, y > 0, x^2 + y^2 < 1)$$

where c is a constant.
(a) [8 marks] Prove or disprove X and $Z = Y^2/(1 - X^2)$ are independent.
(b) [4 marks] Write a definition of X and Y are identically distributed.
(c) [8 marks] Prove or disprove X and Y are identically distributed.
(d) [8 marks] Compute conditional expectation of Y given $X = x$.

Problem 6. Consider a sequence of random variables X_n having densities $f_n(x) = \lambda_n e^{-\lambda_n |x|}/2$ for $x \in \mathbb{R}$ and $\lambda_n > 0$. It is known that $\lambda e^{-\lambda|x|} \to 0$ as $\lambda \to \infty$ or $\lambda \to 0$ for any $x \neq 0$.
(a) [8 marks] When $\lambda_n \to \infty$, show that there exists a random variable X such that $X_n \longrightarrow X$ in distribution as $n \to \infty$. Compute the mean and variance of X.
(b) [8 marks] When $\lambda_n \to 0$, show that there exists no random variable Y such that $X_n \longrightarrow Y$ in distribution as $n \to \infty$.

Useful Information

1. Probability mass/density functions

Distribution	pmf/pdf	domain	mgf
Bernoulli(p)	$p^x(1-p)^{1-x}$	$x = 0, 1$	$1 - p + pe^t$
$N(\mu, \sigma^2)$	$(2\pi\sigma^2)^{-1/2}\exp(-\frac{(x-\mu)^2}{2\sigma^2})$	$x \in \mathbb{R}$	$e^{t\mu + t^2\sigma^2/2}$
Poisson(μ)	$e^{-\mu}\mu^x/x!$	$x = 0, 1, 2, \ldots$	$e^{\mu(e^t-1)}$
beta(α, β)	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$0 < x < 1$	
binomial(n, p)	$\binom{n}{x}p^x(1-p)^{n-x}$	$x = 0, 1, \ldots, n$	$(1 - p + pe^t)^n$
exponential(λ)	$\lambda e^{-\lambda x}$	$x > 0$	$\lambda/(\lambda - t)$
gamma(α, β)	$(\Gamma(\alpha))^{-1}\beta^\alpha e^{-\beta x}x^{\alpha-1}$	$x > 0$	$(1 - t/\beta)^{-\alpha}$

2. The gamma function is defined by $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\, dx$ for $\alpha > 0$ and satisfies $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$.

3. The beta function is defined by beta$(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\, dx$ for $\alpha, \beta > 0$ which satisfies $\beta(\alpha, \beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha + \beta)$.

Total Marks = 100.

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