UNIVERSITY OF TORONTO

Faculty of Arts and Science

AUGUST 2015 EXAMINATIONS

STA347H1S

Duration - 2 hours

Examination Aids: A non-programmable calculator

Last name: ______________________________

First name: ______________ **Middle name:** ______________

Student number: ______________________________

Instruction

1. If a question asks you to so some calculations or derivations, you must show your work to receive full credit. Simplify expressions as much as you can.
2. If you do not have enough space, use the other side and refer it correctly.
3. Some problems could be wrong. If it happens, correct questions by yourself after arguing why the problem is wrong.
4. Please do not pull the pages apart unless necessary. If you do so, print your name and sign all the pages.
5. You can use either pen or pencil for the test. **But please be aware that you are not allowed to dispute any credit after the test is returned if you use pencil.**
6. You are allowed to use any theorem covered in class as well as any statement in other questions in this exam. For example, in question x (y), you can use the result of question x (z) without a proof.
7. There are 4 questions on 3 sheets, and the total is 100 marks. Please hand in your exam paper and books when the time is up.
8. Some useful **information** are on page 3.

Problem 1 (15 marks)**.** Answer the following two questions concerning about almost sure convergence.
(a) [5 marks] Write a definition of almost sure convergence.
(b) [10 marks] A sequence of random variables $X, X_1, X_2, \ldots$ satisfies $\sum_{n=1}^{\infty} P(|X_n - X| > \varepsilon_n) < \infty$ for a sequence ε_n tending to 0. Show that $X_n \longrightarrow X$ almost surely.

Problem 2 (20 marks)**.** Let X and Y be two independent standard normal random variables.
(a) [8 marks] Find the moment generating function of X^2.
(b) [8 marks] Find the moment generating function of XY.
(c) [4 marks] Prove or disprove that X^2 and XY have the same distribution.

Problem 3 (40 marks)**.** Dr Blut found a medicine names "Untere" which drops blood pressures of hypertension patients. To verify the efficiency of new drug Dr Blut recruited n patients and measured blood pressures before taking the medicine X_i and after taking the medicine Y_i for $i = 1, \ldots, n$. Dr Blut found that $X_1, \ldots, X_n \sim F$ and $P(Y_i \le y) = F(y + \nu)$ for $i = 1, \ldots, n$ where F is a distribution function with mean μ and variance σ^2. Assume $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ are independent.
(a) [8 marks] Compute the mean and variance of Y_n.
(b) [8 marks] Show that $\bar{X}_n = (X_1 + \cdots + X_n)/n$ converges to μ either in distribution or in probability or in L^1 or in L^2 or a.s. Pick one of appropriate convergence modes and prove the picked mode.
(c) [8 marks] Find a random variable V_n that converges to σ^2 in probability.
(d) [8 marks] Prove that $\bar{X}_n - \bar{Y}_n$ converges to ν almost surely.
(e) [8 marks] Show that $\sqrt{n/V_n}(\bar{X}_n - \bar{Y}_n - \nu) \xrightarrow{d} N(0, \sigma_e^2)$ and find the σ_e^2.

Problem 4 (25 marks)**.** A new light fast internet service is on testing. It starts working from unstable speed (U). As time goes its speed changes to slow (S) or normal (N) or light fast (F). The status of the internet speed follows a Markov chain having transition matrix

$p =$		**S**	**U**	**N**	**F**
	S	1.0	0.0	0.0	0.0
	U	0.1	0.2	0.6	0.1
	N	0.0	0.0	0.7	0.3
	F	0.0	0.0	0.4	0.6

(a) [4 marks] Determine class (transient or recurrent) of each state.
(b) [7 marks] Find the expected time to exit from status U.
(c) [7 marks] Compute the probability the internet speed drops down to slow eventually.
(d) [7 marks] Compute the limit transition probability $\lim_{n\to\infty} p^{(n)}(U, x)$ for each $x = S, U, N, F$.

Useful Information

1. Probability mass/density functions

Distribution	pmf/pdf	domain	mgf
Bernoulli(p)	$p^x(1-p)^{1-x}$	$x = 0, 1$	$1 - p + pe^t$
Binomial(n, p)	$\binom{n}{x} p^x (1-p)^{n-x}$	$x = 0, \dots, n$	$(1 - p + pe^t)^n$
Poisson(μ)	$e^{-\mu}\mu^x/x!$	$x = 0, 1, 2, \dots$	$e^{\mu(e^t-1)}$
Geometric(p)	$p(1-p)^{x-1}$	$x = 1, 2, 3, \dots$	$pe^t/(1-(1-p)e^t)$
NegBinomial(k, p)	$\binom{x-1}{k-1} p^k (1-p)^{x-k}$	$x = k, k+1, \dots$	$[pe^t/(1-(1-p)e^t)]^k$
Uniform(a, b)	$1/(b-a)$	$a < x < b$	
Exponential(λ)	$\lambda e^{-\lambda x}$	$x > 0$	$(1 - t/\lambda)^{-1}$
Gamma(α, β)	$(\Gamma(\alpha))^{-1}\beta^\alpha e^{-\beta x} x^{\alpha-1}$	$x > 0$	$(1 - t/\beta)^{-\alpha}$
Beta(α, β)	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)/\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$	$0 < x < 1$	
$N(\mu, \sigma^2)$	$(2\pi\sigma^2)^{-1/2}\exp(-\frac{(x-\mu)^2}{2\sigma^2})$	$x \in \mathbb{R}$	$e^{t\mu + t^2\sigma^2/2}$

2. The gamma function is defined by $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\, dx$ for $\alpha > 0$ and satisfies $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$.

3. The sum of geometric sequence is $\sum_{n=0}^{\infty} ab^n = a/(1-b)$ if $|b| < 1$.

4. The abbreviation 'i.i.d.' indicates 'independent and identically distributed.'