UNIVERSITY OF TORONTO

Faculty of Arts and Science

AUGUST 2014 EXAMINATIONS

STA347H1S

Duration - 3 hours

Examination Aids: None

Last name: ______________________________

First name: ______________ **Middle name:** ______________

Student number: ______________________________

Instruction

1. If a question asks you to so some calculations or derivations, you must show your work to receive full credit. Simplify expressions as much as you can.
2. If you do not have enough space, use the other side and refer it correctly.
3. Some problems could be wrong. If it happens, correct questions by yourself after arguing why the problem is wrong.
4. Please do not pull the pages apart unless necessary. If you do so, print your name and sign all the pages.
5. You can use either pen or pencil for the test. **But please be aware that you are not allowed to dispute any credit after the test is returned if you use pencil.**
6. You are allowed to use any theorem covered in class as well as any statement in other questions in this exam. For example, in question x (y), you can use the result of question x (z) without a proof.
7. There are 7 questions on 3 sheets, and the total is 100 marks. Please hand in your exam paper and books when the time is up.
8. Some useful **information** are on page 3.

Problem 1 (10 marks). Let $X_1, X_2, \ldots$ be a sequence of random variables satisfying $\mathbb{E}(X_i) = \mu$, $\mathrm{Cov}(X_i, X_j) = 0$ for $i \neq j$, and $\mathbb{V}\mathrm{ar}(X_i) \leq M$ for all i for some $M > 0$. Show that the sample mean $\overline{X}_n = (X_1 + \cdots + X_n)/n$ converges to μ in L^2.

Problem 2 (20 makrs). Let $X_1, X_2, \ldots$ be i.i.d. from Uniform$(\theta - 1/2, \theta + 1/2)$.
(a) [12 marks] Show that $\bar{X}_n = (X_1 + \cdots + X_n)/n$, $Y_n = 1/2 + X_{(1)} = 1/2 + \min(X_1, \ldots, X_n)$ and $Z_n = -1/2 + \max(X_1, \ldots, X_n)$ converges to θ in probability.
(b) [3 marks] Show that $\sqrt{n}(\bar{X}_n - \theta)$ converges to a Gaussian distribution. Please specify the mean and variance of the limit distribution.
(c) [5 marks] Show that $W_n = n(Y_n - Z_n)$ converges to Gamma$(2, 1)$ in distribution.

Problem 3 (20 marks). A brand new machine starts working from unstable status (U). As time goes it brakes down (B) or gets into either normal working status (N) or super good working status (S). The status of the machine follows a Markov chain having transition matrix

$$p = \begin{array}{c|cccc} & \mathbf{B} & \mathbf{U} & \mathbf{N} & \mathbf{S} \\ \mathbf{B} & 1 & 0 & 0 & 0 \\ \mathbf{U} & 0.1 & 0.6 & 0.2 & 0.1 \\ \mathbf{N} & 0 & 0 & 0.8 & 0.2 \\ \mathbf{S} & 0 & 0 & 0.3 & 0.7 \end{array}$$

(a) [4 marks] Determine class (transient or recurrent) of each state
(b) [6 marks] Find the expected time to exit from status U.
(c) [5 marks] Compute the probability the machine breaks down eventually.
(d) [5 marks] Compute the limit transition probability $\lim_{n\to\infty} p^{(n)}(U, x)$ for each $x = B, U, N, S$.

Problem 4 (5 marks). Suppose a random variable X has mean μ and variance 0. Show that $X = \mu$ almost sure.

Problem 5 (10 marks). Let $Y = Z_1 Z_2 + Z_3 Z_4$ where $Z_i \sim i.i.d.\ N(0, 1)$.
(a) [5 marks] Find the moment generating function of $X = Z_1 Z_2$.
(b) [5 marks] Find the fourth moment of Y.

Problem 6 (30 marks). Let $x_1, x_2, \ldots$ be a sequence of real numbers satisfying $(x_1^k + \cdots + x_n^k)/n \to m_k > 0$ for $k = 2, 4$. Let $\varepsilon_1, \varepsilon_2, \ldots$ be i.i.d. with mean zero, variance one and finite fourth moment. Define $Y_n = \beta x_n + \epsilon_n$ for $n = 1, 2, \ldots$.
(a) [5 marks] Show that $\hat{\beta}_n = \mathrm{argmin}_\beta \sum_{k=1}^n (Y_k - \beta x_k)^2$ solves $\hat{\beta}_n = \sum_{k=1}^n x_k Y_k / \sum_{k=1}^n x_k^2$.
(b) [5 marks] Show that $(x_1\varepsilon_1 + \cdots + x_n\varepsilon_n)/n \to 0$ in probability.
(c) [5 marks] Show that $\hat{\beta}_n \to \beta$ in probability.
(d) [5 marks] Show that $s_n^{-4} \sum_{k=1}^n \mathbb{E}[(x_k\varepsilon_k)^4] \to 0$ where $s_n^2 = \sum_{k=1}^n \mathbb{V}\mathrm{ar}(x_k\varepsilon_k)$.
(e) [5 marks] Conclude that $(x_1\varepsilon_1 + \cdots + x_n\varepsilon_n)/s_n \to N(0, 1)$ in distribution.
(f) [5 marks] Find the limit distribution of $\sqrt{n}(\hat{\beta}_n - \beta)$.

Problem 7 (5 marks)**.** Let $X_1, X_2, \ldots$ be an i.i.d. random variables having probability density function f. Let g be another probability density function. Assume $f(x), g(x) > 0$ for all $x \in \mathbb{R}$. Show that $Y_n = \prod_{k=1}^n \frac{g(X_k)}{f(X_k)}$ is a martingale, that is, $\mathbb{E}[Y_{n+1} \,|\, Y_1, \ldots, Y_n] = Y_n$ for all $n \geq 1$.

Useful Information

1. The standard normal distribution $N(0,1)$ has density function $\phi(z) = (2\pi)^{-1/2} \exp(-z^2/2)$ and moment generating function $\mathrm{mgf}(t) = \exp(t^2/2)$.
2. The density function of Uniform(a, b) for $a < b$ is $1(a < x < b)/(b - a)$.
3. The density and moment generating functions of Gamma(α, β) distribution are $\mathrm{pdf}(x) = 1(x > 0)\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\beta x)$ and $\mathrm{mgf}(t) = (1 - t/\beta)^{-\alpha}$.
4. The sum of geometric sequence is $\sum_{n=0}^\infty ab^n = a/(1 - b)$ if $|b| < 1$.
5. The abbreviation 'i.i.d.' indicates 'independent and identically distributed.'