UNIVERSITY OF TORONTO

Faculty of Arts and Science

AUGUST 2016 EXAMINATIONS

STA347H1S

Duration - 2 hours

Examination Aids: A non-programmable calculator

Last name:

First name: **Middle name:**

Student number:

Instruction

1. If a question asks you to do some calculations or derivations, you must show your work to receive full credit. Simplify expressions as much as you can.
2. If you do not have enough space, use the other side and refer it correctly.
3. Some problems could be wrong. If it happens, correct questions by yourself after arguing why the problem is wrong.
4. Please do not pull the pages apart unless necessary. If you do so, print your name and sign all the pages.
5. You can use either pen or pencil for the test. **But please be aware that you are not allowed to dispute any credit after the test is returned if you use pencil.**
6. You are allowed to use any theorem covered in class as well as any statement in other questions in this exam. For example, in question x (y), you can use the result of question x (z) without a proof.
7. There are 7 questions on 3 sheets, and the total is 100 marks. Please hand in your exam paper and books when the time is up.
8. Some useful **information** are on page 3.

Problem 1. Suppose that X is a random variable of which moment generating function is $\mathrm{mgf}_X(t) = (1 + 3e^{-t} + e^t + e^{2t})/6$ for $-\infty < t < \infty$.
(a) [10 marks] Find the mean and variance of X.
(b) [5 marks] Compute $P(X > 0)$.

Problem 2. [10 marks] Two independent random variables X and Y satisfy $X \sim \mathrm{gamma}(8, 4)$ and $X + Y \sim \mathrm{gamma}(10, 4)$. Find the characteristic function of Y.

Problem 3. [15 marks] Two random variables X and Y satisfy $\mathbb{E}(X \mid Y) = 20 + 0.8Y$ and $\mathbb{E}(Y \mid X) = 16 + 0.1X$. Compute $\mathbb{E}(X)$, $\mathbb{E}(Y)$ and $\mathrm{Corr}(X, Y)$ when X and Y have finite second moments.

Problem 4. Two random variables X and Y have joint probability density function given by

$$\mathrm{pdf}_{X,Y}(x, y) = cxy1(0 < x < y < 1).$$

(a) [5 marks] Find the value of constant c.
(b) [5 marks] Compute marginal density of X and its expectation.
(c) [5 marks] Prove or disprove X and Y have the same distribution.
(d) [5 marks] Prove or disprove X and Y are independent.

Problem 5. New hypothesis is propose in the process of a storm development. A storm is generated as unstable status (U). As time goes its status changes to disappear (D) or normal storm (N) or strong storm (S) or remains unstable. The status of the development process follows a Markov chain having transition matrix

$$p = \begin{array}{c|cccc} & \mathbf{D} & \mathbf{U} & \mathbf{N} & \mathbf{S} \\ \mathbf{D} & 1.0 & 0.0 & 0.0 & 0.0 \\ \mathbf{U} & 0.2 & 0.4 & 0.3 & 0.1 \\ \mathbf{N} & 0.0 & 0.0 & 0.8 & 0.2 \\ \mathbf{S} & 0.0 & 0.0 & 0.6 & 0.4 \end{array}$$

(a) [5 marks] Determine whether or not each state is transient or recurrent.
(b) [5 marks] Compute the limit probability of each status when it started unstable status, that is, $\lim_{n\to\infty} p^{(n)}(U, s)$ each state $s = D, U, N, S$.
(c) [5 marks] Compute the expected time for U absorbed into recurrent states.

Problem 6. Suppose $Y_i \sim$ *ind.* $\mathrm{Poisson}(\mu_i)$ where $\mu_i = \beta x_i$ where β is an unknown positive number and x_i's are known positive numbers satisfying $\lim_{n\to\infty}(x_1^k + \cdots + x_n^k)/n \to m_k > 0$ for $k = 1, 2, 3, 4$. Let $\widehat{\beta} = \mathrm{argmax}_\beta \prod_{i=1}^n \mathrm{pmf}_{Y_i}(y_i)$.
(a) [5 marks] Show that $\widehat{\beta} = \sum_{i=1}^n Y_i / \sum_{i=1}^n x_i$.
(b) [5 marks] Show that $\widehat{\beta}$ converges to β in probability.
(c) [5 marks] Show that $\sqrt{n}(\widehat{\beta} - \beta)$ converges to $N(0, \sigma^2)$ in distribution. State σ^2 using β and m_k's.

Problem 7. [10 marks] Pair-wise independent random variables $X_1, X_2, \ldots$ satisfy $\mathbb{E}(X_n) = \mu$ for all n and $\lim_{n\to\infty} \mathbb{V}\text{ar}(X_n) = 0$. Show that $\overline{X}_n = (X_1 + \cdots + X_n)/n \longrightarrow \mu$ in distribution.

Useful Information

1. Probability mass/density functions

Distribution	pmf/pdf	domain	mgf
Bernoulli(p)	$p^x(1-p)^{1-x}$	$x = 0, 1$	$1 - p + pe^t$
binomial(n, p)	$\binom{n}{x} p^x (1-p)^{n-x}$	$x = 0, \ldots, n$	$(1 - p + pe^t)^n$
Poisson(μ)	$e^{-\mu}\mu^x/x!$	$x = 0, 1, 2, \ldots$	$e^{\mu(e^t-1)}$
geometric(p)	$p(1-p)^{x-1}$	$x = 1, 2, 3, \ldots$	$pe^t/(1-(1-p)e^t)$
negbinomial(k, p)	$\binom{x-1}{k-1} p^k (1-p)^{x-k}$	$x = k, k+1, \ldots$	$[pe^t/(1-(1-p)e^t)]^k$
uniform(a, b)	$1/(b-a)$	$a < x < b$	
exponential(λ)	$\lambda e^{-\lambda x}$	$x > 0$	$(1 - t/\lambda)^{-1}$
gamma(α, β)	$(\Gamma(\alpha))^{-1}\beta^\alpha e^{-\beta x} x^{\alpha-1}$	$x > 0$	$(1 - t/\beta)^{-\alpha}$
beta(α, β)	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)/\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$	$0 < x < 1$	
$N(\mu, \sigma^2)$	$(2\pi\sigma^2)^{-1/2}\exp(-\frac{(x-\mu)^2}{2\sigma^2})$	$x \in \mathbb{R}$	$e^{t\mu + t^2\sigma^2/2}$

2. The gamma function is defined by $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\ dx$ for $\alpha > 0$ and satisfies $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$.

3. The sum of geometric sequence is $\sum_{n=0}^{\infty} ab^n = a/(1-b)$ if $|b| < 1$.

4. The abbreviation 'ind.' indicates 'independent.'

5. If $\lim_{n\to\infty} x_n = x$, then $(x_1 + \cdots + x_n)/n \to x$ as $n \to \infty$.