Fall Term, 2019

Name: ______________ **Surame:** ______________

Section: ____

WILFRID LAURIER UNIVERSITY

Waterloo, Ontario

Mathematics 121 – Introduction to Mathematical Proofs

Midterm 1 – October 11, 2019

Instructors: *Dr. R. Rundle & Dr. P. Zhang*

SOLUTIONS

Time Allowed: *80 minutes*

Total Value: *90 marks*

Number of Pages: *4 plus cover page*

Instructions:

No calculators are allowed. *No other aids are allowed.*

Check that your test paper has no missing, blank, or illegible pages.

Answer in the spaces provided. ***Please note that questions are printed on both sides of the page.***

Show all your work. ***Insufficient justification will result in a loss of marks.***

Student Number: ____________________

[*5 marks*] **1.** Let $U = \{a, b, c, d, e\}$ be the universal set. Let $A = \{a, b, c\}$, $B = \{b, c, d\}$ and $C = \{c, d, e\}$. Determine:

(a) $A \cup (B \cap C) = \boxed{\{a, b, c, d\}}$

(b) $A \cup B \cup C = \boxed{\{a, b, c, d, e\} = U}$

(c) $A^c \cap B^c \cap C^c = \boxed{\emptyset}$

(d) $(A - (A \cup B)) \cup \{\emptyset\} = \boxed{\{\emptyset\}}$

(e) $(B - A) - (\emptyset - \{\emptyset\}) = \boxed{\{d\}}$

[*7 marks*] **2.** (a) Complete the following truth tables.

p	q	$p\wedge \sim q$	$\sim p \wedge q$	$p \to q$	$\sim (q \to p)$	$p \to\sim q$	$\sim q \to\sim p$	$q \to\sim p$	$\sim (q\wedge \sim q)$
T	T	F	**F**	**T**	F	F	T	F	T
T	F	T	F	F	**F**	T	F	**T**	T
F	T	**F**	T	T	T	T	**T**	T	**T**
F	F	F	F	T	F	**T**	T	T	T

[*4 marks*] (b) According to the truth tables above in part (a), determine whether each of the following statements is TRUE or FALSE. Circle your answer.

1) $(q \to\sim p) \Leftrightarrow (p \to\sim q)$ ([TRUE] / FALSE)   2) $(\sim p \wedge q) \Leftrightarrow (p\wedge \sim q)$ (TRUE / [FALSE])

3) $\sim p \wedge q \Leftrightarrow\sim (q \to p)$ ([TRUE] / FALSE)   4) $(\sim q \to\sim p) \Leftrightarrow (p \to q)$ ([TRUE] / FALSE)

[*4 marks*] **3.** (a) Complete the truth table of the compound statement with the the disjunctive normal form

$(p\wedge q\wedge r)\vee(p\wedge q\wedge \sim r)\vee(p\wedge \sim q\wedge \sim r)\vee(p\wedge \sim q\wedge r)\vee(\sim p\wedge q\wedge r)\vee(\sim p\wedge q\wedge \sim r)\vee(\sim p\wedge \sim q\wedge \sim r)$.

p	q	r	statement
T	T	T	T
T	T	F	**T**
T	F	T	**T**
T	F	F	T
F	T	T	T
F	T	F	**T**
F	F	T	**F**
F	F	F	T

[*3 marks*] (b) Simplify the statement in part (a) to an **equivalent** compound **conditional** statement that has the simplest expression in which each of p, q and r appears **only once**.

Answer: $\sim (\sim p\wedge \sim q \wedge r) \Leftrightarrow p \vee q\vee \sim r \Leftrightarrow r \to (p \vee q)$

4. Let the set of real numbers $\mathbb{R}$ be the universal set for x. Let the set of natural numbers $\mathbb{N}$ be the universal set for K. Let $p(x, K)$ denote the mathematical statement

$$-K \leq f(x) < K.$$

(a) Convert the following quantified logical statement **S** into a **mathematical** statement.

$$\mathbf{S}: \quad \forall K \ [\ \exists x \ [\sim p(x, K)]\ ]$$

The mathematical statement that the statement **S** represnts is :

[*4 marks*] **Answer**: *For each $K \in \mathbb{N}$, there exists an $x \in \mathbb{R}$ such that $f(x) < -K$ or $f(x) \geq K$.*

(b) Write a logical statement that is equivalent to the **negation** of the quantified statement **S**, in which the connective $\sim$ disappears.

[*3 marks*] **Answer**: $\sim [\ \forall K\ [\ \exists x\ [\sim p(x, K)]\ ]\ ] \iff \exists K\ [\sim [\ \exists x\ [\sim p(x, K)]\ ]\ ] \iff \exists K\ [\ \forall x\ [p(x, K)]\ ]$

(c) Translate the final answer in part (b) into a **mathematical** statement.

[*3 marks*] **Answer**: *There exists a $K \in \mathbb{N}$ such that for all $x \in \mathbb{R}$, $-K \leq f(x) < K$.*

[*10 marks*] **5**. Let S be the following statement in set theory:

$$\textit{If } A = B, \textit{ then } A - B = \emptyset.$$

Complete the following sentences.

(a) S is a ___ $\boxed{TRUE}$ ___ (TRUE or FALSE) statement.

(b) The **converse** of the statement S is:

________________ $\boxed{If\ A - B = \emptyset,\ then\ A = B}$ ________________ .

(c) The **converse** of the statement S is a ___ $\boxed{FALSE}$ ___ (TRUE or FALSE) statement.

(d) The **contrapositive** of S is a ___ $\boxed{TRUE}$ ___ (TRUE or FALSE) statement, which is

________________ $\boxed{If\ A - B \neq \emptyset,\ then\ A \neq B}$ ________________ .

(e) The **negation** of the statement S is a ___ $\boxed{FALSE}$ ___ (TRUE or FALSE) statement, which ___ $\boxed{IS}$ ___ (IS or IS NOT) equivalent to the statement:

$$A = B \textit{ but there is some } x \in A \cap B^c.$$

(f) The fact that

$$\mathbb{N} \subset \mathbb{Z} \textit{ and so } \mathbb{N} - \mathbb{Z} = \emptyset$$

is $\boxed{\textit{a counterexample}}$ (an **example** or a **counterexample**) for the **converse** statement of S.

(g) The example in part (f) ________ $\boxed{PROVES}$ ________ (PROVES, DISPROVES or GIVES NO CONCLUSION ABOUT) the **negation** of the **converse** statement of S.

(h) The **converse** statement of the **contrapositive** of S ___ $\boxed{HAS}$ ___ (HAS or HAS NO) **conterexamples**.

[*9 marks*] **6**. Use the Principle of **Mathematical Induction** to prove that, for all natural numbers $n > 5$,

$$2^n > (n+1)^2.$$

Proof. The base case: Let $n = 6$.

$$2^6 = 64 > 49 = (6+1)^2.$$

Assume that for some $k \geq 6$, the inequality holds; that is,

$$2^k > (k+1)^2.$$

It then follows that for $n = k+1$,

$$2^{k+1} = 2 \cdot 2^k > 2(k+1)^2 \quad \text{(by the inductive assumption above)}$$

$$= 2k^2 + 4k + 2 = (k^2 + 4k + 4) + k^2 - 2$$

$$> (k+2)^2 \qquad \text{(as } k \geq 6 \text{ and hence } k^2 - 2 > 0\text{)}$$

The inequality holds also for $n = k+1$.

Thus the mathematical induction completes the proof.

7. Let $n \in \mathbb{Z}$. Prove the statement

If $3n-5$ is even, then n is odd

using the following **three different methods**.

[*4 marks*] (a) (*Direct proof*)

Proof. Suppose that $3n-5$ is even; that is, $3n-5=2k$, where $k \in \mathbb{Z}$.

It follows that $n = 2k-2n+5 = 2k-2n+4+1 = 2(k-n+2)+1$ is odd because $k-n+2 \in \mathbb{Z}$.

[*5 marks*] (b) (*Proof of the contrapositive*)

Proof. The contrapositive of the statement is:

If n is even, then $3n-5$ is odd.

Assume that n is even; that is, $n = 2k$, where $k \in \mathbb{Z}$.

It follows that $3n-5 = 3(2k)-5 = 2(3k-3)+1$ is odd because $3k-3 \in \mathbb{Z}$.

[*6 marks*] (c) (*Indirect proof by contradiction*)

Proof. Assume that $3n-5$ is even, and n is also even.

This implies that $n = 2k$ for some $k \in \mathbb{Z}$ and $3n-5 = 2m$ for some $m \in \mathbb{Z}$.

It follows from $2m = 3n-5 = 3(2k)-5 = 2(3k-3)+1$ that $2(m-3k+3) = 1$, which is even because $m-3k+3 \in \mathbb{Z}$.

This contradiction shows that the statement "if $3n-5$ is even, then n is odd" is true.

[*7 marks*] **8**. Let a, d be real numbers such that $a < d$. Prove the statement

There exist $b, c \in \mathbb{R}$ such that $a < b < c < d$

by a **constructive proof**.

[Hint: You **must** use the assumption that $a < d$, but **cannot** use any specific values of a or d in your proof.]

Proof. Let $b = \frac{2}{3}a + \frac{1}{3}d$ and $c = \frac{1}{3}a + \frac{2}{3}d$.

Then

$$a = \frac{2}{3}a + \frac{1}{3}a < \frac{2}{3}a + \frac{1}{3}d = b < \frac{1}{3}a + \frac{2}{3}d = c < \frac{2}{3}d + \frac{1}{3}d = d.$$

because $a < d$ implies that $2a + d < a + 2d$.

An alternative proof: Let $b = \frac{1}{2}(a+d)$.

Then

$$a = \frac{1}{2}(a+a) < \frac{1}{2}(a+d) = b < \frac{1}{2}(d+d) = d.$$

Similarly, for $b < d$, we may take $c = \frac{1}{2}(b+d)$, and verify that $b < c < d$.

Putting them together we know that

$$a < b < c < d.$$

9. **Prove** or **disprove** the following statements. Write **all necessary steps**. In each case, indicate first **which method** from the following four methods you are using:

1) *Direct proof of implication*
2) *Proof of the contrapositive of implication*
3) *Indirect proof of implication by contradiction*
4) *Disproof of implication by counterexample*

[*4 marks*] (a) Let A and B be nonempty sets.

If $A \neq B$, *then* $(A - B) \cup (B - A) = A \cup B$.

The method: __ **4** __

Let $A = \{1, 2\}$ and let $B = \{2, 3\}$.
Obviously $A \neq B$, and $A \cup B = \{1, 2, 3\}$.
Because $A - B = \{1\}$ and $B - A = \{3\}$, $(A - B) \cup (B - A) = \{1, 3\}$.
Since $(A - B) \cup (B - A) \neq A \cup B$, we have a counterexample for the statement.

[*5 marks*] (b) Let m and n be integers.

If $m^2 \geq n^2$, *then* $m - n \geq 0$ *or* $m + n < 0$.

The method: __ **4** __

A counterexample: Let $m = -1$ and $n = 1$.
Then $m^2 = (-1)^2 = 1 = 1^2 = n^2$, so the condition $m^2 \geq n^2$ is satisfied.
But $m - n = -1 - 1 = -2 < 0$, so the conclusion $m - n \geq 0$ fails.
Also $m + n = -1 + 1 = 0$, so the other possible conclusion $m + n < 0$ also fails.

Partial marks will be given to the follwoing trial:
The method: __ **2** __

Proof. The contrapositive: *If* $m - n < 0$ *and* $m + n \geq 0$, *then* $m^2 < n^2$.
Assume that $m - n < 0$ and $m + n \geq 0$.
Then $m^2 - n^2 = (m - n)(m + n) \leq 0$.
This implies that $m^2 \leq n^2$.
This does not prove the contrapositive statement as the conclusion should be the strict inequality $m^2 < n^2$.

[*6 marks*] (c) Let A, B be sets.

If $A \cup B \subseteq A \cap B$, *then* $A = B$.

The method: ___ **1** ___

Suppose that $A \cup B \subseteq A \cap B$.
Then $A \subseteq A \cup B \subseteq A \cap B \subseteq B$, and $B \subseteq A \cup B \subseteq A \cap B \subseteq A$.
Therefore $A = B$.