Fall Term, 2018

Name: ____________ **Last Name:** ____________

Section: ____

WILFRID LAURIER UNIVERSITY

Waterloo, Ontario

Mathematics 121 – Introduction to Mathematical Proofs

Midterm 2 – November 17, 2018

Instructors:

Dr. P. Zhang: Section A - 10:00 a.m.

Dr. R. Rundle: Sections B & C - 11:30 a.m. & 5:30 p.m.

Time Allowed: *80 minutes*

Total Value: *90 marks*

Number of Pages: *4 plus cover page*

Instructions:

No calculators are allowed. *No other aids are allowed.*

Check that your test paper has no missing, blank, or illegible pages.

Answer in the spaces provided. ***Please note that questions are printed on both sides of the page.***

Show all your work. ***Insufficient justification will result in a loss of marks.***

SOLUTIONS

Student Number: ____________

[*8 marks*] **1.** Complete the following sentences by filling the blanks:

(a) The **Addition Counting Principle** says that if A and B are finite sets with $|A| = m$, $|B| = n$ and $A \cap B = \boxed{\emptyset}$, then

$$|\boxed{A \cup B}| = m + n.$$

(b) Let A, B be subsets of a finite set S with $A \cap B = \emptyset$. Denote by $\mathcal{P}(X)$ the power set of X. Then $\boxed{T}$ is the **truth value** of the statement

$$|\mathcal{P}(A)| + |\mathcal{P}(B)| - 1 = |\mathcal{P}(A) \cup \mathcal{P}(B)|.$$

(c) We can apply the **Pigeonhole Principle** to prove the following statement.

If we choose five distinct numbers from $\{1, 2, 3, 4, 5, 6, 7, 8\}$*, there must be a pair of numbers in the selected five with a sum of 9.*

The "pigeonholes" are $\boxed{\{1, 8\}}$, $\boxed{\{4, 5\}}$, $\{3, 6\}$, $\{2, 7\}$ and the number of "pigeons" is $\boxed{5}$.

(d) Let A be an **infinite** set. Consider the statement

Every *one-to-one function* $h : A \to A$ *is also onto.*

The **truth value** of the statement is $\boxed{F}$.

(e) Let A and B be finite sets with $|A| = m$, $|B| = n$ and $|A \cap B| = k$. The number of elements in the product set $(A - B) \times (B - A)$ is $\boxed{(m - k)(n - k)}$.

[*10 marks*] **2.** Consider the set U of **50 students** in the MA121 class with the following data:

25 students are **also** taking MA122;

The subset B of U consists of **35** students who are **also** taking MA103;

The subset C of U consists of **10** students who are **also** taking MA104;

15 students in the class are taking **both** MA122 and MA103;

MA103 and MA104 **cannot** be taken at the same time;

5 students in the class are taking **both** MA122 and MA104.

In the following questions, either **fill the blanks** or **circle the correct answers**.

(a) The number of elements of the set $A = \{x \in U : x \text{ takes MA122}\}$ is $\boxed{\mathbf{25}}$.

(b) The students who are also taking both MA103 and MA104 form a set called the $\boxed{\textbf{empty}}$ set.

(c) The number of elements of the set $A \cap B \cap C$ is $|A \cap B \cap C| = \boxed{\mathbf{0}}$.

(d) The number of students who are taking MA121 and **at least another** course listed here at the same time form the set $A \cup \boxed{\mathbf{B \cup C}}$. The number of elements of this set can be computed by the formula

$$|A| + |B| + |C| \boxed{-} |A \cap B| - |\boxed{\mathbf{A \cap C}}| - |B \cap C| \boxed{+} |A \cap B \cap C|.$$

(e) The number of elements of the set defined in part (d) is

(1) 70

(2) 20

$\boxed{\textbf{(3) 50}}$

(f) The students in this class who are also taking **exactly two more** courses listed here form the set

(1) $A \cup B \cup C$

(2) $(A \cap B) \cup (A \cap C) \cup C$

$\boxed{\textbf{(3) } \mathbf{A \cap (B \cup C)}}$

(g) The number of students in this class who are **not** taking any other course listed here is

(1) 5

$\boxed{\textbf{(2) 0}}$

(3) 50

Over

3. Let n and k be **nonnegative** integers.

[*4 marks*] (a) Fill the **four** blanks in the following formula so that the identity holds:

$$\sum_{r=\boxed{1}}^{k} (-1)^r \binom{k}{r-1} = \sum_{j=0}^{\boxed{k-1}} (-1)^{\boxed{j+1}} \binom{k}{j} = \boxed{-1} \sum_{j=0}^{k-1} (-1)^j \binom{k}{j}$$

[*5 marks*] (b) Fill the **five** blanks in the following formula so that the identities hold:

$$\sum_{r=\boxed{0}}^{\boxed{k+1}} (-1)^r \binom{k+1}{r} = 1+\sum_{r=\boxed{1}}^{k} (-1)^r \binom{k+1}{r} + (-1)^{k+1} \binom{k+1}{\boxed{k+1}} = 1+\sum_{r=1}^{k} (-1)^r \binom{k+1}{r} + (-1)^{\boxed{k+1}}$$

[*10 marks*] (c) Prove that, for every **positive** integer n,

$$\sum_{r=0}^{n} (-1)^r \binom{n}{r} = 0$$

by **mathematical induction**, using the identities in parts (a) and (b) as well as the identity $\binom{k+1}{r} = \binom{k}{r} + \binom{k}{r-1}$, where $k \geq r \geq 1$.

[**Warning**: Do **NOT** apply the binomial theorem!]

Proof. The Base Case: For $n = 1$, we verify that

$$\sum_{r=0}^{1} (-1)^r \binom{1}{r} = (-1)^0 \binom{1}{0} + (-1)^1 \binom{1}{1} = 1 \times 1 - 1 \times 1 = 0.$$

Assume that the identity holds for some $n = k \geq 1$; that is,

$$\sum_{r=0}^{k} (-1)^r \binom{k}{r} = 0.$$

For $n = k+1$, We get

$$\begin{aligned}
\sum_{r=0}^{k+1} (-1)^r \binom{k+1}{r} &= (-1)^0 \binom{k+1}{0} + \sum_{r=1}^{k} (-1)^r \binom{k+1}{r} + (-1)^{k+1} \binom{k+1}{k+1} \\
&= 1 + \sum_{r=1}^{k} (-1)^r \left[\binom{k}{r} + \binom{k}{r-1} \right] + (-1)^{k+1} \\
&= \left[(-1)^0 \binom{k}{0} + \sum_{r=1}^{k} (-1)^r \binom{k}{r} \right] + \left[\sum_{r=1}^{k} (-1)^r \binom{k}{r-1} + (-1)^{k+1} \binom{k}{k} \right] \\
&= \left[\sum_{r=1}^{k} (-1)^r \binom{k}{r} \right] + \left[\sum_{j=0}^{k-1} (-1)^{j+1} \binom{k}{j} + (-1)^{k+1} \binom{k}{k} \right] \\
&= 0 + \sum_{j=0}^{k} (-1)^{j+1} \binom{k}{j} \\
&= 0 - \sum_{j=0}^{k} (-1)^j \binom{k}{j} \\
&= 0 - 0 = 0.
\end{aligned}$$

This completes the induction, and hence the formula holds for all positive integers n.

[*8 marks*] **4**. Find and simplify the term containing y^{104} in the binomial expansion of $(x^2y + xy^2)^{102}$. Show **all** your work.

Solution. The k-th term in the expansion is $\binom{102}{k} (x^2y)^{102-k}(xy^2)^k = \binom{102}{k} x^{204-k}y^{102+k}$

The sought term contains y^{104}, so it is the term that makes $102 + k = 104$; that is, $k = 2$.

Thus the sought term is $\binom{102}{2} x^{204-2}y^{102+2} = \frac{102!}{2!100!} x^{202}y^{104} = 5151x^{202}y^{104}$

5. **Prove** each of the following statements **using ONLY the specified method**.

[*7 marks*] (a) Let A, B be subsets of a universal set U. Denote by $\mathscr{P}(X)$ the power set of the set X. Use the method of the **indirect proof of implication by contradiction** to prove the statement:

$$\textit{If } \mathscr{P}(A) \subseteq \mathscr{P}(B), \textit{ then } A \subseteq B.$$

Proof. Assume that $\boxed{\mathscr{P}(A) \subseteq \mathscr{P}(B)}$ and $\boxed{A \not\subseteq B}$.

It follows from $A \not\subseteq B$ that $\underline{A \notin \mathscr{P}(B)}$.

But $A \in \mathscr{P}(A)$.

So it follows from $\mathscr{P}(A) \subseteq \mathscr{P}(B)$ that $\underline{A \in \mathscr{P}(B)}$.

This is a contradiction, which shows that the statement

$$\textit{if } \mathscr{P}(A) \subseteq \mathscr{P}(B), \textit{ then } A \subseteq B$$

is true.

[*6 marks*] (b) Use only the **formulas** $n! = n(n-1)!$ and $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ to **prove directly** the statement:

If m is a **positive** *integer, then m divides* $\binom{m^2}{m}$.

Proof.
$$\binom{m^2}{m} = \frac{m^2!}{m!(m^2-m)!}$$
$$= \frac{m^2(m^2-1)!}{m(m-1)!(m^2-m)!}$$
$$= m \cdot \frac{(m^2-1)!}{(m-1)!(m^2-m)!}$$
$$= m\binom{m^2-1}{m-1}$$

Because $m - 1 \geq 0$ as m is positive, $\binom{m^2-1}{m-1}$ is defined as a binomial coefficient, which is an integer.

Therefore $m | \binom{m^2}{m}$.

[*6 marks*] (c) Let $n \in \mathbb{N}$. Prove the statement:

$$\textit{If } n \neq 1, \textit{ then } n \nmid (n! - 1).$$

by proving its **contrapositive statement**.

Proof. The contrapositive statement is: *If $n|(n!-1)$, then $n = 1$.*

Suppose that $n|(n!-1)$.

Then $n! - 1 = kn$ for some $k \in \mathbb{N}$.

Thus $n! - kn = 1$.

It follows from $n|kn$ and $n|n!$ that $n|1$.

This implies that $n = 1$.

[*12 marks*] **6**. (a) Use **Euclid's Algorithm** to find the greatest common divisor d of 374 and -946.

$$946 = 2 \times 374 + 198$$
$$374 = 1 \times 198 + 176$$
$$198 = 1 \times 176 + 22$$
$$176 = 8 \times 22 + 0$$

So $\gcd(374, -946) = \gcd(946, 374) = 22$.

(b) Find **integers** k and l such that $d = \gcd(374, -946) = 374k - 946l$.

$$22 = 198 + (-1) \times 176$$
$$176 = 374 + (-1) \times 198$$
$$198 = 946 + (-2) \times 374$$

Thus

$$\begin{aligned} 22 &= 198 + (-1) \times 176 \\ &= 198 + (-1) \times [374 + (-1) \times 198] = 2 \times 198 + (-1) \times 374 \\ &= 2 \times [946 + (-2) \times 374] + (-1) \times 374 \\ &= (-5) \times 374 + 2 \times 946 \\ &= (-5) \times 374 - (-2) \times 946 \end{aligned}$$

Therefore we may take $k = -5$ and $l = -2$ such that $22 = \gcd(374, -946) = 374k - 946l$.

[*14 marks*] **7**. Given a function $f : S \to S$, where S is a finite set. Let $x \in S$ and $y \in S$ be the variables for the following predicate statements in **propositional logic**:

$$p(x, y) : \ x = y \ ; \qquad q(x, y) : \ f(x) = f(y) \ ; \qquad r(x, y) : \ f(x) = y \ .$$

(a) Write a **mathematical** statement that is implied **directly** by the following statement in propositional logic:

$$\forall x \forall y \ [\sim p(x, y) \to \sim q(x, y)].$$

Answer: *For any $x, y \in S$, if $x \neq y$, then $f(x) \neq f(y)$.*

(b) Convert the definition of an onto function $f : S \to S$ into a **quantified** statement in **propositional logic** in which the quantifiers $\forall$ and $\exists$ both appears once.

Answer: $\forall y \ [\exists x \ [r(x, y)]]$

(c) Write the **negation** of the **logical statements** in parts (a) and (b) and **simplify** them until the connective $\sim$ appears inside the brackets $[\ \cdots\ ]$ only once in each statement.

Answer: Negation of the **logical** statement given in (a):

$$\begin{aligned} & \sim (\forall x \forall y \ [\sim p(x, y) \to \sim q(x, y)]) \\ \iff & \exists x \exists y \ [\sim ((\sim (\sim p(x, y)) \vee (\sim q(x, y))] \\ \iff & \exists x \exists y \ [(\sim p(x, y)) \wedge q(x, y)] \end{aligned}$$

Answer: Negation of the **logical** statement found in (b):

$$\begin{aligned} & \sim (\forall y \ [\exists x \ [r(x, y)]]) \\ \iff & \exists y \ [\sim (\exists x \ [r(x, y)])] \\ \iff & \exists y \ [\forall x \ [\sim r(x, y)]] \end{aligned}$$