Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

Calculus I for the Life Sciences MAT 1330E

Midterm 2

November 13, 2019

Prof. Andrew Wagner

You must **sign below** to confirm that you have read, understand, and will follow these **instructions**:

- This is a **75-minute** closed-book exam; no notes are allowed. You may use a calculator if it has no differentiation, integration, or graphing capabilities. Otherwise, no electronics, no notes, scrap paper, cell phones, smartwatches or related devices of any kind are permitted. All such devices, including cell phones, **must be stored in your bag under your desk or at the front or back of the room for the duration of the exam**.
- The exam consists of 10 questions on 9 pages. If you need additional space for rough work, use the backs of the pages. If there is not enough room for your solution in the space provided, then use the back of the previous page and clearly indicate where you have done so. **Do not detach any of the pages.**
- Questions 1 through 6 are multiple choice, worth a total of 6 points. **Record your answers to the multiple choice questions in the table in the middle of page 2.**
- Questions 7–10 are long answer, worth points as indicated. **You must show your work, your work must be legible and well-justified, and you must record your answers in the spaces provided.**
- Good luck! (: ← Version C

LAST NAME: ________________

First name: ________________

Student Number: ________________

Signature: ________________

SOLUTIONS

Please circle the DGD section where you would like to have your test returned to you:

DGD 1	DGD 2	DGD 3
[illegible] 8:30 – 9:50	WED 10:00 – 11:20	WED 13:00 – 14:20
VNR 1095	VNR 3075	MRN 021

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For grading purposes only:

Question	1–6	7	8	9	10	Total
Max	6	3	2	3	6	20
Marks						

Please enter your multiple choice answers in the boxes below.

Question	1	2	3	4	5	6
Answer	E	D	A	C	B	F

Multiple-choice questions:
Only your final answer will be considered. **Enter your multiple-choice answers in the table on the previous page.**

1. (1 point) Suppose f and g are two differentiable functions. Some values of $f(x)$, $f'(x)$, $g(x)$ and $g'(x)$ (for $x = 2, 3, 5$) are given in the table below.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
2	5	-3	3	5
3	1	7	5	-1
5	-1	3	2	5

If $h(x) = f(g(x))$, what is the value of $h'(5)$?

A. 15

B. 35

C. -5

D. 21

E. -15

F. -30

$h'(x) = f'(g(x))g'(x)$

$h'(5) = f'(g(5))g'(5)$

$= f'(2) \cdot 5$

$= -3 \cdot 5 = -15$

2. (1 point) If $g(x) = \sqrt{e^{-2x} + 2x^2}$, then $g'(x)$ is

A. $\dfrac{1}{2\sqrt{-2e^{2x} + 4x}}$

B. $(-2e^{-2x} + 4x)\sqrt{e^{-2x} + 2x^2}$

C. $\dfrac{e^{-2x} + 2x^2}{2\sqrt{e^{-2x} + 2x^2}}$

D. $\dfrac{-e^{-2x} + 2x}{\sqrt{e^{-2x} + 2x^2}}$

E. $-e^{-x} + \sqrt{2}$

F. $\dfrac{1}{2\sqrt{e^{-2x} + 2x^2}}$

$$g'(x) = \overbrace{\frac{1}{2\sqrt{e^{-2x}+2x^2}}}^{\frac{d}{dx}\sqrt{\cdot}} \cdot \overbrace{(-2e^{-2x} + 4x)}^{\text{because of chain rule}}$$

, then divide everything by 2.

$\frac{d}{dx}e^{-2x}$ $\quad$ $\frac{d}{dx}2x^2$

3. (1 point) Which of the following is the derivative of $g(x) = \ln(e^{-2x}x^6 \sin(x)6^x)$?

A. $-2 + \frac{6}{x} + \cot(x) + \ln(6)$ **B.** $\frac{-2e^{-2x} + 6x^5 + \cos(x) + 6^x \ln(6)}{e^{-2x}x^6 \sin(x)6^x}$ **C.** $\frac{-2e^{-2x} + \cos(x)}{e^{-2x}x^6 \sin(x)6^x}$

D. $-1 + \cot(x)$ **E.** $-2 + \frac{6}{x} + \tan(x)$ **F.** $-2 + \ln(6) + 6x^5 + \sec(x)$

$g(x) = \ln(e^{-2x}) + \ln x^6 + \ln(\sin x) + \ln(6^x)$

$g'(x) = \frac{1}{e^{-2x}} \cdot -2e^{-2x} + \frac{1}{x^6} \cdot 6x^5 + \frac{1}{\sin x} \cdot \cos x + \frac{1}{6^x} \cdot 6^x \ln 6$

$= -2 + \frac{6}{x} + \cot x + \ln 6$

You can also differentiate right away, but it's looong.

4. (1 point) Consider the curve implicitly defined by the equation

$$y^3 + y^2 = x^2 + 3.$$

What is the slope of the tangent line to the curve at the point $(3, 2)$?

A. $\frac{1}{8}$ **B.** $\frac{2}{7}$ **C.** $\frac{3}{8}$ **D.** $\frac{1}{7}$ **E.** $\frac{1}{5}$ **F.** $\frac{1}{16}$

Differentiate both sides wrt x:

$3y^2\frac{dy}{dx} + 2y\frac{dy}{dx} = 2x$

$\Rightarrow \frac{dy}{dx}(3y^2 + 2y) = 2x$

$\Rightarrow \frac{dy}{dx} = \frac{2x}{3y^2+2y}$

So $\left.\frac{dy}{dx}\right|_{(3,2)} = \frac{2\cdot 3}{3\cdot 2^2 + 2\cdot 2} = \frac{6}{12+4} = \frac{3}{8}$

5. (1 point) If $f(x) = x^{(3^x)}$ for $x > 0$, then $f'(x) =$

A. $x^{3^x-1}3^x\ln(3)$ **B.** $x^{3^x}3^x\left(\ln(3)\ln(x) + \frac{1}{x}\right)$ **C.** $x^{3^x}\left(\ln(x) + \frac{1}{x}\right)$

D. $x^{3^x}3^x\ln(3)\ln(x)$ **E.** $x^{3^x}3^x\ln(3)$ **F.** $x^{3^x}3^x(\ln(3) + \ln(x))$

$\ln f(x) = \ln(x^{(3^x)}) = 3^x \ln x$

Implicit differentiation:

$$\frac{f'(x)}{f(x)} = 3^x \ln 3 \ln x + 3^x \cdot \frac{1}{x}$$

$$\Rightarrow f'(x) = f(x) \cdot 3^x(\ln 3 \ln x + \tfrac{1}{x}) = x^{(3^x)} \cdot 3^x(\ln 3 \ln x + \tfrac{1}{x})$$

6. (1 point) Given that the Taylor polynomial of order 4 of a function f centred at the base point $a = 1$ is

$$T_4(x) = 3 + (x-1)^2 + \frac{1}{12}(x-1)^3 + \frac{1}{12}(x-1)^4$$

which one of the following statements is correct?

A. $f(1) = 3$, $f'''(1) = \frac{1}{4}$ and $f^{(4)}(1) = \frac{1}{2}$ **B.** $f(1) = 3$, $f'''(1) = 2$ and $f^{(4)}(1) = 2$

C. $f(1) = \frac{1}{3}$, $f'''(1) = \frac{1}{2}$ and $f^{(4)}(1) = \frac{1}{2}$ **D.** $f(1) = 3$, $f'''(1) = \frac{1}{3}$ and $f^{(4)}(1) = 6$

E. $f(1) = \frac{1}{2}$, $f'''(1) = \frac{1}{6}$ and $f^{(4)}(1) = 3$ **F.** $f(1) = 3$, $f'''(1) = \frac{1}{2}$ and $f^{(4)}(1) = 2$

$$T_4(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f^{(3)}(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4$$

$\Downarrow$

$f(1) = 3$

$\Downarrow$

$\frac{f^{(3)}(1)}{6} = \frac{1}{12}$

$\Rightarrow f^{(3)}(1) = \frac{1}{2}$

$\Downarrow$

$\frac{f^{(4)}(1)}{24} = \frac{1}{12}$

$\Rightarrow f^{(4)}(1) = 2$

Long-answer questions: You must show your work, your work must be legible and well-justified, and you must record your answers in the spaces provided.

7. (3 points) Using the definition of the derivative (i.e., from first principles), determine the derivative of $f(x) = \sqrt{x-5}$. Make sure you start by writing the definition and use proper mathematical notation throughout.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{(x+h)-5} - \sqrt{x-5}}{h}$$

$$= \lim_{h \to 0} \left(\frac{\sqrt{x+h-5} - \sqrt{x-5}}{h}\right)\left(\frac{\sqrt{x+h-5} + \sqrt{x-5}}{\sqrt{x+h-5} + \sqrt{x-5}}\right)$$

$$= \lim_{h \to 0} \frac{(x+h-5) - (x-5)}{h\left(\sqrt{x+h-5} + \sqrt{x-5}\right)}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h-5} + \sqrt{x-5}\right)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h-5} + \sqrt{x-5}}$$

$$= \frac{1}{\sqrt{x+0-5} + \sqrt{x-5}}$$

$$= \frac{1}{2\sqrt{x-5}}$$

8. (2 points) Using methods from calculus and algebra, evaluate the following limit if it exists; otherwise, justify mathematically why it does not exist. **Make sure you identify any indeterminate forms you encounter and any theorems that you use from class.**

$$\lim_{x\to 0} \frac{e^x \sin(3x) - 3x}{3x^2}$$ ← indeterminate form of type $\frac{0}{0}$

*l'Hopital's Rule

$$= \lim_{x\to 0} \frac{e^x \sin(3x) + e^x(\cos(3x))(3) - 3}{6x}$$

$$= \lim_{x\to 0} \frac{e^x(\sin(3x) + 3\cos(3x)) - 3}{6x}$$ ← indet. form of type $\frac{0}{0}$

*l'Hopital's Rule

$$= \lim_{x\to 0} \frac{e^x(\sin(3x) + 3\cos(3x)) + e^x(3\cos(3x) - 9\sin(3x)) - 0}{6}$$

$$= \lim_{x\to 0} \frac{e^x(-8\sin(3x) + 6\cos(3x))}{6} = \frac{6}{6} = 1$$

* provided the new limit exists or is $\pm\infty$

9. (3 points) Let $f(x) = 2x^4 - 16x^2 + 7$ for $-3 \le x \le 3$. Find the absolute maximum and minimum of f on this interval, and the values of x at which f attains these extreme values.

Absolute maximum value $\boxed{25}$ attained at the point(s) $x = \boxed{\pm 3}$

Absolute minimum value $\boxed{-25}$ attained at the point(s) $x = \boxed{\pm 2}$

Justify your work below!

$f'(x) = 8x^3 - 32x$

$0 = 8x^3 - 32x$

$0 = 8x(x^2 - 4)$

$x = 0 \qquad x = \pm 2$

$f(0) = 7$

$f(-2) = 2(-2)^4 - 16(-2)^2 + 7 = -25$

$f(2) = -25$

ABS. MIN.

$f(-3) = 25$

$f(3) = 25$

ABS. MAX

10. (6 points) Our goal for this question is to produce the graph of $f(x) = \dfrac{-x}{x^3-1}$.

We provide you with the 1st and 2nd derivatives of f below, which may be useful:

$$f'(x) = \frac{1+2x^3}{(x^3-1)^2} \quad \text{and} \quad f''(x) = \frac{-6x^2(x^3+2)}{(x^3-1)^3}$$

(a) Write the domain of f. $\boxed{\{x \in \mathbb{R} : x \neq 1\}}$

(b)Determine the critical number(s) of f (round your answers to 3 decimal places), or write 'none' if there aren't any.

$$0 = \frac{1+2x^3}{(x^3-1)^2} \Rightarrow 0 = 1+2x^3 \Rightarrow x^3 = -\tfrac{1}{2} \Rightarrow x = \sqrt[3]{-\tfrac{1}{2}} \approx -0.794$$

Critical number(s) of f: $\boxed{-0.794}$

(c) Determine the intervals where f is increasing/decreasing. Identify any local minimum/maximum points. Justify your answers using the first derivative, and write your findings in a clearly organized table.

VA $x=1$

	$(-\infty, \sqrt[3]{-\frac{1}{2}})$	$(\sqrt[3]{-\frac{1}{2}}, 1)$	$(1, \infty)$
Sign of f'	$-$	$+$	$+$
behaviour of f	DEC	INC	INC

f has a local min. @ $x = \sqrt[3]{-\frac{1}{2}}$

(d)Determine the inflection point candidate(s) of f (round your answers to 3 decimal places), or write 'none' if there aren't any.

$$0 = \frac{-6x^2(x^3+2)}{(x^3-1)^3} \Rightarrow 0 = -6x^2(x^3+2)$$

$x=0$, $x = \sqrt[3]{-2} \approx -1.260$

IP candidate(s) of f: $\boxed{0 \text{ and } -1.260}$

(e) Determine the intervals where f is concave up/down. Identify any inflection points. Justify your answers using the second derivative, and write your findings in a clearly organized table.

	$(-\infty, \sqrt[3]{-2})$	$(\sqrt[3]{-2}, 0)$	$(0, 1)$	$(1, \infty)$
Sign of f''	$-$	$+$	$+$	$-$
behaviour of f	CONCAVE DOWN	CONCAVE UP	CONCAVE UP	CONCAVE DOWN

IP @ $x = \sqrt[3]{-2}$

(f) Recall that $f(x) = \frac{-x}{x^3-1}$. Determine all vertical and/or horizontal asymptotes of f, if any. Show your work.

$$\lim_{x\to\infty} \frac{-x}{x^3-1} = \lim_{x\to\infty} \frac{\cancel{x^3}\left(-\frac{1}{x^2}\right)}{\cancel{x^3}\left(1-\frac{1}{x^3}\right)} = \lim_{x\to\infty} \frac{\left(-\frac{1}{x^2}\right)}{1-\frac{1}{x^3}} = \frac{0}{1-0} = 0$$

$$\lim_{x\to-\infty} \frac{-x}{x^3-1} = 0$$

$\therefore f$ has H.A. $y=0$ as $x\to\infty$ and as $x\to-\infty$

$$\lim_{x\to 1^-} \frac{-x}{x^3-1} \begin{matrix} \rightsquigarrow -1 \\ \rightsquigarrow 0^- \end{matrix} = \infty$$

$$\lim_{x\to 1^+} \frac{-x}{x^3-1} \begin{matrix} \rightsquigarrow -1 \\ \rightsquigarrow 0^+ \end{matrix} = -\infty$$

$\therefore f$ has a V.A. @ $x=1$.

(g) Using your answers to (a)–(f), and any further values or limits you need, sketch the graph of $y = f(x)$ on the grid given below, and label all important points found in parts (a)–(f).