

Practice Problem Set #1 Fall 2019

Both our human lives and the outworking of most terrestrial energy systems takes place within the thin layer of atmosphere and oceans found on the “spinning surface of a circling sphere” (McLeish). We don’t always think about this reality, but we’ll explore it a little here, where we’ll use a few basic facts and relations to derive others.

Multiple Choice

1. How far then can a 2m tall human standing on the top of the CN tower (400m) see to a clear horizon? (Hint 1: You are being asked to calculate the distance to the perceived horizon as a function of an observer’s height h . Assume a flat landscape such as the Ocean. Hint 2: Triangles and the Earth’s radius).

- a. 722 km b. 5.1 km **c. 72.2 km** d. 723.6 km
e. 1000 km

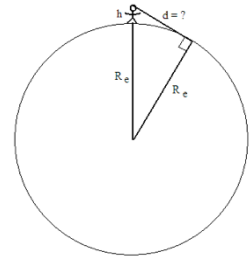
Using Pythagorean theorem: $a^2 + b^2 = c^2$

$$d^2 + R_e^2 = (h + R_e)^2$$

$$d^2 + R_e^2 = h^2 + 2hR_e + R_e^2$$

$$d^2 = h^2 + 2hR_e$$

$$d = \sqrt{h^2 + 2hR_e}$$



Helpful Simplification: Note in the expression for d that the first h^2 term under the square root sign is negligible for typical values of d (i.e., $h \ll R$). This gives a very handy approximation: $d = \sqrt{2hR_e} = \sqrt{hD_e} = 3.6 \sqrt{h}$ if h is in metres (m) and the distance d is in kilometres (km).

This handy formula is an easy thing to “carry around with you” to be able to estimate, and visualize, earth horizon distances in a variety of “near Earth” contexts.

It is a helpful way to remind yourself of the curvature of the Earth, something that is surprising easy to disregard. There are endless and important implications of the Earth’s shape and motion to terrestrial energy systems.

Note:

$h = 402$ m (400m for the CN Tower + 2m individual).

$R_e = 6,371$ km $\rightarrow \therefore D_e = 2R_e \rightarrow D_e = 2 \times 6371\text{km} = 12742$ km

Solution: $d = 3.6 \sqrt{h} = 3.6 \sqrt{402} = 72.2$ km

2. At what distance from you should a dime (with assumed diameter of 1.8 cm) be placed such that it exactly covers the Sun?

- a. 1.9 m b. 2.1 m c. 2.0 m d. 3.0 m e. 19 m

Solution:

$$D_s = 1.39 \times 10^9 \text{ m}$$

$$d_s = 150 \times 10^9 \text{ m}$$

$$D_d = 0.018 \text{ m}$$

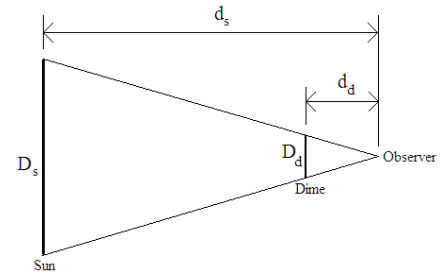
By similar triangles:

$$\frac{D_s}{d_s} = \frac{D_d}{d_d}$$

$$d_s \quad d_d$$

$$\frac{1.39 \times 10^9 \text{ m}}{150 \times 10^9 \text{ m}} = \frac{0.018 \text{ m}}{d_d}$$

$$d_d = 1.9 \text{ m}$$



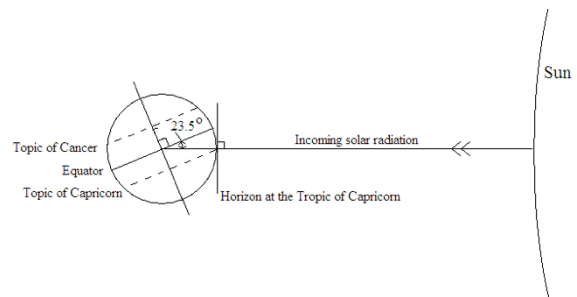
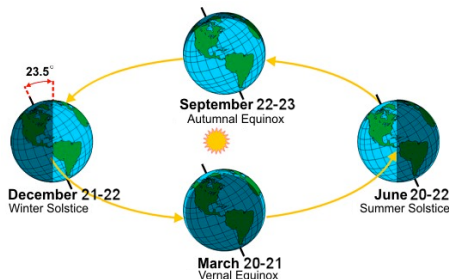
3. Toronto is found at approximately 44.5 degrees north latitude. (i) How far above the horizon does the sun achieve at solar noon on the shortest day of the year in the north hemisphere (21 Dec.) (Hint: see the insert on page 43 of the text book or the lecture notes.)?

- a. 90° b. 18° c. 20° d. 22° e. 78°

Solution:

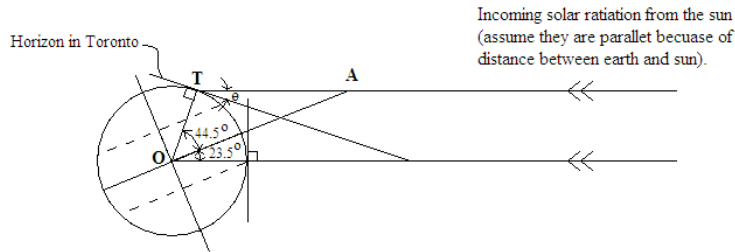
Recall two things: first that there is always some part of the Earth directly facing the sun, and that the Tropics of Capricorn (23.5°S) is the most southerly latitude at which the sun can appear directly overhead (90° above the horizon) at solar noon on December 21.

This is when the southern hemisphere is tilted towards the sun at its maximum extent.



Thus, if you have good intuition you can argue that every degree that you are north or south of the latitude where the sun is directly overhead will shift the sun a degree less than 90°. This allows direct calculation of the required angles. Thus, at 60°N, you are (60°+23.5° = 83.5° degrees from this, and the sun is only 6.5° above the horizon at solar noon.

Mathematically, if you let northern latitudes be positive, and southern negative, then the angle is 90° – abs (latitude where the sun is directly overhead – current latitude).



Using geometry

1. Extend equator to the incoming solar radiation to form point A.
2. Label Toronto T and the center of the earth O.
3. $\text{OAT} = 23.5^\circ$ (parallel lines, Z)
4. $\text{OTA} = 180^\circ - 44.5^\circ - 23.5^\circ = 112^\circ$ (Triangle)
5. $\theta = 112^\circ - 90^\circ = 22^\circ$

4. Assuming a solar insolation of exactly 1370 W/m^2 how much total energy does the Earth receive from the sun?

- a. $1.7 \times 10^{17} \text{ W}$ b. 1.7 W c. $2.0 \times 10^{17} \text{ W}$ d. $3.5 \times 10^{17} \text{ W}$
 e. $8.5 \times 10^{16} \text{ W}$

Solution: (Which is also in the lecture notes, Lectures 1-3, Slides 32-33)

Step 1: Determine the projected area of the earth

$$SA_p = \pi R_e$$

$$SA_p = \pi(6.35 \times 10^6 \text{ m})^2$$

$$SA_p = 1.3 \times 10^{14} \text{ m}^2$$

Step 2: Determine the amount of energy received over this area

$$E_e = (\text{SI})(SA_p) \dots \text{ where } SI \sim \text{solar insolation (in their lecture notes)}$$

$$E_e = (1370 \text{ w/m}^2)(1.3 \times 10^{14} \text{ m}^2)$$

$$E_e = 1.74 \times 10^{17} \text{ W or } 170 \text{ PW}$$

This is a good number for basic reference and worth remembering: the Earth receives a total of about 170 PW from the sun.

5. Calculate the average orbital speed (in m/s) of the Earth around the sun.

- a. $6.0 \times 10^4 \text{ m/s}$ b. $1.5 \times 10^4 \text{ m/s}$ c. $3.0 \times 10^4 \text{ m/s}$ d. $7.5 \times 10^3 \text{ m/s}$
 e. $1.2 \times 10^5 \text{ m/s}$

Solution: Assume earth's path around the sun is circular (which is indeed an excellent approximation).

$$L = 2\pi R_{o_e}$$

$$L = 2\pi(150 \times 10^9 \text{ m})$$

$$L = 9.42 \times 10^{11} \text{ m}$$

Determine the duration of the orbit (365 days): $t = 365 \times 86400 = 31,536,000\text{s}$

(Note: $86400 = 60\text{s/min} \times 60\text{min/hr} \times 24 \text{ hr/day}$)

The average orbital speed: $v = L/T$

$$v = (9.42 \times 10^{11} \text{ m}) / (31,536,000 \text{ s})$$

$$v = 3.0 \times 10^4 \text{ m/s}$$

6. What is the rotational velocity of Earth in m/s on the equator?

a. 463 m/s

b. 1666.8 m/s

c. 833.4 m/s

d. 231.5 m/s

e. 926 m/s

Solution:

At the equator earth's diameter is $12.7 \times 10^6 \text{ m}$

Step 1: The earth's circumference is: $L = \pi D = \pi(12.7 \times 10^6 \text{ m}) = 4.0 \times 10^7 \text{ m}$

Step 2: Determine the duration of one rotation over a day: $t = 86400 \text{ s}$

Rotational velocity: $v = L/t$

$$v = (4.0 \times 10^7 \text{ m}) / (86400 \text{ s})$$

$$v = 463 \text{ m/s}$$

Short Answer

1. Let's go back to MC #2, the question with the dime...

i) At what distance from you should a dime (with assumed diameter of 1.8 cm) be placed such that it exactly covers the moon?

Solution:

$$\frac{D_m}{d_m} = \frac{D_d}{d_d}$$

$$\frac{3.48 \times 10^6 \text{ m}}{4.0 \times 10^8 \text{ m}} = \frac{0.018 \text{ m}}{d_d}$$

$$d_d = 2.1 \text{ m}$$

ii) What does this imply about the nature of a total eclipse?

Solution: Since the distances are almost equal, when a total eclipse occurs the moon just blocks the sun, but leaves only the outer most part so we can see solar flares happening.

iii) If the Earth were modeled as a large globe of 1 m diameter, how thick would a proportional representation be of the average ocean depth (4 km)? (Look up the relevant diameters or check the lectures for them).

Solution: The diameter of the Earth is $1.27 \times 10^7 \text{ m}$. So using a ratio the model heights can be determined.

The average depth of the ocean:

$$\frac{1 \text{ m}}{1.27 \times 10^7 \text{ m}} = \frac{h_{od}}{4000 \text{ m}}$$

$h_{mo} = 3.15 \times 10^{-4} \text{ m} = 0.315 \text{ mm}$ → Which is about the thickness of three average pieces of paper

iv) Again, if the Earth were modeled as a large globe of 1 m diameter, how thick would a proportional representation be of the average troposphere thickness (12 km)?

The average thickness of the troposphere:

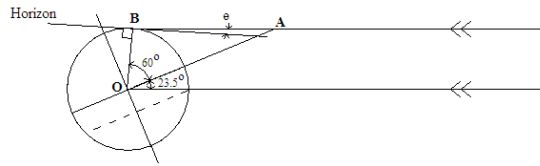
$$\frac{1 \text{ m}}{1.27 \times 10^7 \text{ m}} = \frac{h_{tt}}{12000 \text{ m}}$$

$$h_{mo} = 9.45 \times 10^{-4} \text{ m} = 0.945 \text{ mm}$$

Note: The trick with this solution is that the troposphere is 3 times the thickness of the oceans. One could simply multiple 0.315 mm by 3 and get the same answer!

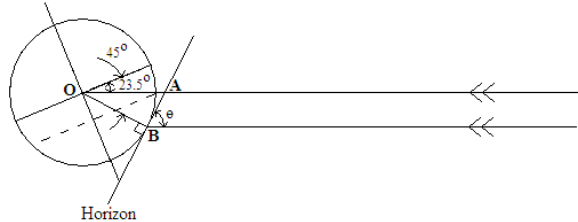
2. Let's go back to MC #3, the question about Toronto's latitude

i) What is the angle of the sun at the equator?
Using geometry or direct reasoning, $\theta = 66.5^\circ$
($=90^\circ$ - the tilt of the earth)



ii) What is the angle of the sun on the Tropic of Capricorn (23.5°S)?

As reasoned above, this angle clearly $\theta = 90^\circ$
(sun directly overhead at noon)



iii) What is the angle of the sun at 60 degrees north latitude?

Using geometry

1. Extend equator to the incoming solar radiation to form point A.
2. $OAB = 23.5^\circ$ (parallel lines, Z)
3. $OBA = 180^\circ - 60^\circ - 23.5^\circ = 96.5^\circ$ (Triangle)
4. $\theta = OBA - 90^\circ = 6.5^\circ$

iv) At 45 degrees south?

(Hint: see the insert on page 43 of the text book or the lecture notes.)

Using geometry

1. $AOB = 45^\circ - 23.5^\circ = 21.5^\circ$
2. $OAB = 180^\circ - 90^\circ - 21.5^\circ = 68.5^\circ$ (Triangle)
3. $\theta = OAB = 68.5^\circ$ (parallel lines, Z)

3. Let's go back to MC #4 on Solar Radiation...

Assuming a solar insolation of exactly 1370 W/m^2 :

i) Approximately what fraction of the Sun's total energy output does the Earth receive?

Solution: The total energy passing through a spherical shell at earth's orbit is

$$E_{oe} = (SI)(4\pi R_{oe}^2)$$

Giving a fraction of:

$$\frac{E_e}{E_{oe}} = \frac{(SI)(\pi R_e^2)}{(SI)(4\pi R_{oe}^2)}$$

$$\frac{E_e}{E_{oe}} = \frac{R_e^2}{4R_{oe}^2}$$

$$\frac{E_e}{E_{oe}} = \frac{(6.36 \times 10^6 \text{ m})^2}{4(150 \times 10^9 \text{ m})^2}$$

$$\frac{E_e}{E_{oe}} = 4.5 \times 10^{-10}$$

E_{oe}

ii) Why is it clear that the Earth radiates almost the same amount of energy as it receives? The Earth radiates almost the same amount of energy as it receives because the temperature of the Earth remains about constant when averaged over its entire surface. Of course, there are important variations of great significance, leading to the possibility of global warming, or at other times the initiation of ice ages, or other departures from expected or "normal" conditions.

iii) Estimate the variation of solar intensity, at the average Earth distance from the sun, between the near side of the Earth and its far side from the sun. According to the inverse-square law, solar intensity at each of these points (Far side and Near side) is inversely proportional to the square of their distance from the sun. Because the surface area of a sphere increases with the square of the radius.

R_{NEO} = Distance from Near side of the Earth to the Sun = $R_{AEO} - R_e$

R_{FEO} = Distance from Far side of the Earth to the Sun = $R_{AEO} + R_e$

R_e = Radius of the Earth

S_{Near} = Solar intensity at Near side of the Earth

S_{Far} = Solar intensity at Far side of the Earth

$$(S_{Near})(4\pi R_{NEO}^2) = (SI)(4\pi R_{AEO}^2) \Rightarrow S_{Near} = \frac{SI(R_{AEO}^2)}{R_{NEO}^2}$$

$$(S_{Far})(4\pi R_{FEO}^2) = (SI)(4\pi R_{AEO}^2) \Rightarrow S_{Far} = \frac{SI(R_{AEO}^2)}{R_{FEO}^2}$$

$$S_{Near} - S_{Far} = SI \left[\frac{R_{AEO}^2}{(R_{AEO} - R_e)^2} - \frac{R_{AEO}^2}{(R_{AEO} + R_e)^2} \right] = SI \left[\frac{1}{\left(1 - \frac{R_e}{R_{AEO}}\right)^2} - \frac{1}{\left(1 + \frac{R_e}{R_{AEO}}\right)^2} \right]$$

$$R_e = 6.35 \times 10^6 \text{ m}$$

$$R_{AEO} = 150 \times 10^9 \text{ m}$$

$$SI = 1370 \text{ W/m}^2$$

$$S_{Near} - S_{Far} = 1370 \left[\frac{1}{\left(1 - \frac{6.35 \times 10^6}{150 \times 10^9}\right)^2} - \frac{1}{\left(1 + \frac{6.35 \times 10^6}{150 \times 10^9}\right)^2} \right] = 1370(1.694 \times 10^{-4}) = 0.23$$

So the difference in intensity between the two sides is about a quarter of a W/m². For most purposes, such a variation is trivial. However, by contrast, the variation over the year, as the Earth comes nearer and then moves further from the sun, is slightly more

significant, but still a minor issue for most purposes (and this difference it is partly compensated for by the difference in the time the Earth spends in these positions.

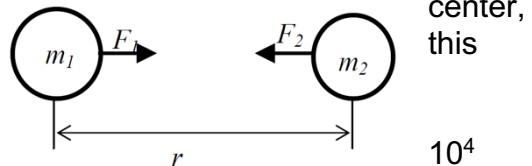
4) Let's go back to MC #5, the Earth's orbital speed...

i) Calculate the average orbital speed in km/h of the Earth around the sun. (Remember you calculated this in m/s above!)

$$3.0 \times 10^4 \text{ m/s} = 3.0 \times 10^4 \times 1000 \text{ [m/km]} / 3600 \text{ [s/hr]} = 1.1 \times 10^5 \text{ km/h}$$

ii) What is its average acceleration toward the and what produced the required force to achieve acceleration?

Determine the average acceleration: $a = v^2 / R_{oe}$
 $a = (3.0 \times 10^4 \text{ m/s})^2 / (150 \times 10^9 \text{ m})$



$$a = 6.0 \times 10^{-3} \text{ m/s}^2$$

This force is achieved by the gravitation attraction between the earth and the sun.

iii) Newton's law of gravitation states that the gravitational attraction between two bodies is proportional to the product of their masses and inversely proportional to the square of the distance separating their centers of mass. The proportionality constant is usually taken as $6.67 \times 10^{-11} \text{ Nm}^2/\text{kg}^2$ Use this law and the fact that an object on the Earth's surface experiences an acceleration of 9.81 m/s^2 to deduce the mass of the Earth.

Newton's Law of gravitation.

$$F = \frac{Gm_1m_2}{r^2} \quad F = F_1 = F_2$$

To determine the mass of the earth, let m_1 be the mass of the falling object and m_2 be the mass of the earth. From Newton's second law the net force on the falling object is

$$F = ma$$

$$\therefore F = m_1g$$

so

$$m_1g = \frac{Gm_1m_2}{R_e^2}$$

$$m_2 = \frac{gR_e^2}{G}$$

$$m_2 = \frac{(9.81 \text{ m/s}^2)(6.35 \times 10^6 \text{ m})^2}{(6.67 \times 10^{-11} \text{ Nm}^2/\text{kg}^2)}$$

$$m_2 = 5.93 \times 10^{24} \text{ kg}$$

iv) Use the acceleration of the Earth in its orbit to deduce the mass of the sun.

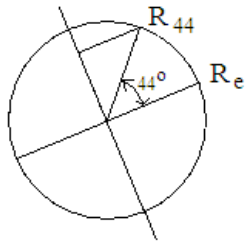
To determine the mass of the sun let m_1 be the mass of sun and m_2 be the mass of the earth; r is the radius of earth's orbit around the sun; $F = m_2a$ where $a = 6.0 \times 10^{-3} \text{ m/s}^2$

$$m_2a = \frac{Gm_1m_2}{R_{oe}^2}$$

$$m_2 = \frac{aR_{oe}^2}{G}$$

$$m_2 = \frac{(6.0 \times 10^{-3} \text{ m/s}^2)(150 \times 10^9 \text{ m})^2}{(6.67 \times 10^{-11} \text{ Nm}^2/\text{kg}^2)}$$

$$m_2 = 2.02 \times 10^{30} \text{ kg}$$



v) Compare your answers with generally accepted values of these two astronomical constants.

The generally accepted value of the earth is very close to the above value:

$$5.98 \times 10^{24} \text{ kg}$$

The generally accepted value of the sun is very close to the above

value:

$$1.99 \times 10^{30} \text{ kg}$$

5) Let's go back to MC #6, the rotational velocity of the earth...

(i) What is the rotational velocity of Earth in km/h on the equator?(Remember you calculated this in m/s above)

$$463 \text{ m/s} \times (1000\text{m/km}) \times (3600 \text{ s/h}) = 1666.8 \text{ km/h}$$

ii) What is the rotational velocity of Earth in m/s and km/h at 44° north latitude?

$$R_{44} = R_e \cos 44^\circ$$

$$R_{44} = (6.35 \times 10^6 \text{ m}) \cos 44^\circ$$

$$R_{44} = 4.57 \times 10^6 \text{ m}$$

$$v_{44} = L/t$$

$$v_{44} = 2\pi R_{44}/t$$

$$v_{44} = 2\pi (4.57 \times 10^6 \text{ m}) / (86400 \text{ s})$$

$$v_{44} = 332.3 \text{ m/s}$$

$$v_{44} = 1196.3 \text{ km/h}$$

iii) What is its average acceleration toward the center on the equator, and what produced the required force to achieve this acceleration?

$$a = v^2/R_e$$

$$a = (463 \text{ m/s})^2 / (6.35 \times 10^6 \text{ m})$$

$$a = 0.034 \text{ m/s}^2$$

The earth's rotation causes the surface to move fastest at the equator and not at all at the poles. This causes two forces, one real, one an apparent force. The real force is centripetal acceleration, the force required to keep us rotating and its source is gravity.

The second, the so-called centrifugal force, is an apparent force and only happens as we're on a rotating Earth and can be thought of 'inertia'. It wants to fling you off at 90deg to the centre of rotation, or in a straight line away from the Earth's surface (not outward). The centrifugal force acts opposite to the force of gravity, in turn reducing it ever so

slightly. However, as the Earth is slightly bulged at the equator, the latter reduces the impact of the centrifugal force.

6) The moon is 384,000 km from the Earth and is a sphere, as we can see from Earth. If the Earth radiates on average 342 Wm^{-2} of energy back to space, what will the moon receive from Earth *on its surface* on average? The Earth has a radius of 6,375 km. Ignore any differences due to varying orbital distances due to the planet's circular nature as you considered in in q.3 (iii) (i.e. treat it as one single orbital distance).

ANS:

$$\begin{aligned}\text{Total energy output of Earth} &= \text{Energy output intensity of Earth} \times 4\pi r_{\text{Earth}}^2 \\ &= 342 \text{ Wm}^{-2} \times 4\pi(6,375,000)^2 \\ &= 1.74 \times 10^{17} \text{ W}\end{aligned}$$

$$\begin{aligned}\text{Energy intensity at the moon's orbital radius} &= 1.74 \times 10^{17} / 4\pi r_{\text{moon's orbit}}^2 \\ &= 1.74 \times 10^{17} \text{ W} / 4\pi(384,000,000)^2 \\ &= 0.094 \text{ Wm}^{-2}\end{aligned}$$

At the surface we have to spread this out over the moon's surface, so we can short cut and divide by 4 to give **0.023 Wm⁻²**.