L14. L'Hôpital's Rule

Recall that when we solve limits, we sometimes get an indeterminate form ($\frac{0}{0}$ or $\frac{\infty}{\infty}$) if we try to substitute.

e.g. $\lim_{x \to 2} \frac{x^2-4}{x-2}$ or $\lim_{x \to \infty} \frac{3x^2+2x-1}{x^2+1}$

We had some techniques for solving some varieties of these problems, but now we can use differentiation.

Thm (L'Hôpital's Rule)

Suppose $f(x)$ and $g(x)$ are diff. fcns and $\lim_{x \to a} \frac{f(x)}{g(x)}$ is indeterminate of type $\frac{0}{0}$ ($\lim_{x \to a} f(x) = 0$, $\lim_{x \to a} g(x) = 0$) or of type $\frac{\infty}{\infty}$ ($\lim_{x \to a} f(x) = \pm\infty$, $\lim_{x \to a} g(x) = \pm\infty$).

If $\lim_{x \to a} g'(x) \neq 0$, then we have

$$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}}$$

provided the RHS limit exists or is $\pm\infty$.

Examples (stuff we already know)

1) $\lim_{x \to 2} \frac{x^2-4}{x-2}$ is solved by factoring, but we can also use L'Hôpital's Rule:

$$\lim_{x \to 2} \frac{x^2-4}{x-2} \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 2} \frac{2x}{1} \quad \text{(L'Hôpital's)}$$

$$= 4.$$

2) $\lim_{x \to 0} \frac{\sqrt{x^2+9}+3}{x^2}$ is solved by rationalizing, but we can use L'H:

$$\lim_{x \to 0} \frac{\sqrt{x^2+9}-3}{x^2} \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{\frac{1}{2\sqrt{x^2+9}} \cdot 2x}{2x} \quad \text{(L'Hôpital's)}$$

$(x+3)(\sqrt{x}+3)$

$$= \frac{1}{2\sqrt{9}} = \frac{1}{6}$$

$6\sqrt{6}$

3) $\lim_{x \to \infty} \frac{3x^2+2x-1}{x^2+1}$ can be solved by looking at degrees and coefficients of the highest powers of x, or by using L'H:

$$\lim_{x \to \infty} \frac{3x^2+2x-1}{x^2+1} \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x \to \infty} \frac{6x+2}{2x} \quad \text{(L'H) - still } \frac{\infty}{\infty}$$

$$= \lim_{x \to \infty} \frac{6}{2} \quad \text{(L'H again)}$$

$$= 3.$$

Essentially, we can use L'Hôpital's Rule instead of remembering how to solve each kind of problem.

Plus, we can solve previously insoluble things.

Example $\lim_{x \to \infty} x^2 \cdot e^{-x}$

If we sub $x \to \infty$, we get $\infty \cdot 0$. Usually, multiplying by ∞ gives ∞, but multiplying by 0 gives 0. So which one wins?

We can use L'Hôpital's Rule, but only if we rearrange it to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ first.

* When we have $0 \cdot \infty$, we should take the reciprocal of one of the factors and put it in the denominator. *

Note $\frac{1}{e^{-x}} = e^x$, so we just write

$$\lim_{x \to \infty} x^2 e^{-x} = \lim_{x \to \infty} \frac{x^2}{e^x} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x \to \infty} \frac{2x}{e^x} \quad (L'H) \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x \to \infty} \frac{2}{e^x} \quad (L'H)$$

$$= 0.$$

Notice that x^2 slowly degrades to a constant, but the e^x persists. This is because e^x grows to ∞ "more quickly" than x^2 does, so the ratio looks like $\frac{\text{small}}{\infty} = 0$.

Example $\lim_{x\to 0^+}\left(\frac{1}{x^2}-\frac{1}{\tan x}\right)$

If we let $x\to 0^+$, we get $\frac{1}{0}-\frac{1}{0}$ i.e. $\infty-\infty$. We don't know if this will be 0, or ∞, $-\infty$, or anything in between.

We can use L'Hôpital's Rule if we bring it over a common denominator.

$$\lim_{x\to 0^+}\left(\frac{1}{x^2}-\frac{1}{\tan x}\right)=\lim_{x\to 0^+}\frac{\tan x - x^2}{x^2\tan x}\quad\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0^+}\frac{\sec^2 x-2x}{2x\tan x+x^2\sec^2 x}=\frac{1}{0}$$

The denominator is $\oplus$ since $2x\tan x>0$ and $x^2\sec^2 x>0$ when $x>0$.

$\therefore$ the limit is $+\infty$.

Example $\lim_{x\to 3^+}\frac{1-e^{x-3}}{(x-3)^2}\quad\left(\frac{0}{0}\right)$

$$=\lim_{x\to 3^+}\frac{-e^{x-3}}{2(x-3)}\ (\text{L'H})=\frac{-1}{0^+}$$ $\Leftarrow$ denom. $\oplus$ since $2(x-3)>0$ when $x>3$.

$$=-\infty.$$

Note, we cannot use L'Hôpital's Rule when it's not $\frac{0}{0}$ or $\frac{\infty}{\infty}$! If we had applied it here, we would get the wrong answer!

$$\lim_{x\to 3^+}\frac{-e^{x-3}}{2(x-3)}\ \text{"="}\ \lim_{x\to 3^+}\frac{-e^{x-3}}{2}=\frac{-1}{2}.\quad \times$$

So L'Hôpital's Rule is powerful, but only if you check if you can use it.

Example $\lim_{x \to 0^+} x^x = 0^0$?? Anything to the exponent 0 is 1 and 0 to any power is 0. What happens here?

We use the same trick we usually do when we have an exponent we want to get rid of.

$$\lim_{x \to 0^+} x^x = e^{\ln\left(\lim_{x \to 0^+} x^x\right)} = e^{\lim_{x \to 0^+} \ln(x^x)}$$ ← we can move lim outside of ln since ln is cts

$$= e^{\lim_{x \to 0^+} x \ln x}$$ → $0 \cdot -\infty$, so we rearrange

$$= e^{\lim_{x \to 0^+} \frac{\ln x}{(1/x)}} \quad \left(\frac{\infty}{\infty}\right)$$

$$= e^{\lim_{x \to 0^+} \frac{1/x}{(-1/x^2)}} \quad \text{(L'Hôpital's)}$$

$$= e^{\lim_{x \to 0^+} (-x)}$$

$$= e^0$$

$$= 1.$$

You can solve this slightly differently if you want.

Let $L = \lim_{x \to 0^+} x^x$

$\Rightarrow \ln L = \ln\left(\lim_{x \to 0^+} x^x\right)$

$= \ldots = 0$

$\Rightarrow L = e^0 = 1.$

Example $\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^x$ (where $k\in\mathbb{R}$ is a constant, $k\neq 0$)

If we sub $x\to\infty$, then this looks like 1^∞. 1 to any power is 1, but any $a>1$ to the power ∞ is ∞. So what happens?

We solve this like we solve 0^0:

$$L=\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^x$$

$$\Rightarrow \ln L = \ln\left(\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^x\right)$$

$$=\lim_{x\to\infty}\ln\left(\left(1+\frac{k}{x}\right)^x\right)$$

$$=\lim_{x\to\infty} x\ln\left(1+\frac{k}{x}\right)$$

$$=\lim_{x\to\infty}\frac{\ln\left(1+\frac{k}{x}\right)}{1/x} \quad \left(\frac{0}{0}\right)$$

$$=\lim_{x\to\infty}\frac{\frac{1}{1+k/x}\cdot\frac{-k}{x^2}}{\frac{-1}{x^2}} \quad \text{(L'Hôpital's)}$$

$$=\lim_{x\to\infty}\frac{-k}{1+k/x}$$

$$=k$$

$$\Rightarrow L=e^k.$$

Note: the expression $\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^x$ is one of the definitions of e^k, found in financial math.

If you have an interest rate of r (e.g. $r=0.02=2\%$), a bank account with \$$P$, and interest is compounded annually, then after 1 year you'll have \$$P\cdot(1+r)$.

After t years, you'll have

$$\$P(1+r)^t = \$P\underbrace{(1+r)\cdot(1+r)\cdot\ldots\cdot(1+r)}_{t \text{ factors}}$$

But if your interest is compounded twice a year, then the interest rate is divided by 2 and applied 2 times as often, so you end up with

$\$P(1+\frac{r}{2})^{2t}$ after t years. $100 turns into $225 after 1 year ...

If you compound it...	you get...	$100 after 1 year @ 100%...
monthly	$\$P(1+\frac{r}{12})^{12t}$	$261.30
daily	$\$P(1+\frac{r}{365})^{365t}$	$271.46
continuously	$\lim_{n\to\infty} \$P\underbrace{(1+\frac{r}{n})^{nt}}$	$271.82

$= \$P(e^r)^t$

$= \$Pe^{rt}$