

Lecture 1

SEPTEMBER 10, 2018

TEXTBOOK NOTES

Numerical concepts are fostered in all human societies and while some ideas are innate, they also appear to vary based on cultural contexts.

1.1 COUNTING THINGS

The earliest record of counting discrete quantities is called *tallying*, using notched, sticks, knotted strings or other artifacts as markers of the objects to be counted (1:1 basis). Ex. the Ishango bone, from the Upper Paleolithic era, was a tally stick representing a lunar calendar.

The purpose of tallying is to record an ongoing enumeration and was used to derive other forms of final numerations (like Roman number signs).

Tallies were largely used for arithmetic calculation and accounting. For example, the Royal Treasurer and Cutter recorded taxpayer loan payments as tallies on wooden sticks, stock sale and other financial transactions.

THE QUIPU

Knots, *quipu*, were widespread among various cultures as ways of recordkeeping and counting. The quipu was made and used by the Incas people in 1250-1560, from cotton and llama fibers and were formed by a primary cord with several pendant cords attached to it. Depending on the complexity of the information, the pendant cords could also have subsidiaries attached to them.

The knotted strings formed a decimal place-value system because each level represents a power of ten (Top to Bottom: 10, 000s to 1s). The quantitative information can be derived from the knots, but it is still widely unknown what the colours, twists and other features of the knots meant. It is believed that the quipus served as instruments of the administration of the state.

The quipu was studied by several individuals (quipucamayucs) during the Inca empire to create a system for a complex society and appears to be similar to the modern numeral system used by our society as well.

The Inca empire was vanquished by Francisco Pizarro. Spanish viceroy Francisco de Toledo worked to create a reformed society among the Inca natives and colony. Part of the reform required the translation of the quipu data into written data by the locals and quipucamayucs.

In the early stages, the Spaniards were respectful of the quipu script when they were establishing the colonial rule; however, it was later prohibited after the Spanish rulers realized it was a threat. It was considered a threat to colonial officials because no Spaniard was able to master the quipu script and had to rely on Inca translators, which remained as a symbol of their autonomy.

THE COUNTING BOARD

Early civilizations developed methods for calculations based on their accessibility to materials and the technologies available to them. The most portable and inexpensive tool for calculating were fingers, *digit* derived from *digitus* which is finger/toe in Latin. Another tool for calculating was the counting board, flat surface with lines on which pebbles were placed to conduct arithmetic operations. The Romans converted the term into *abacus* and the oldest surviving Greek abacus is from the Island of Salamis.

It is considered that the Salamis tablet primarily produced calculations for money transactions because of the lines on it resembling the Greek currency and the illustrations on the Darius vase, images of the treasurer using the counting board to record tributes (sac of coins) made to the king.

Different cultures had similar devices to perform specific calculation techniques. The Romans used a hand-held abacus with counters similar to the Greek Salamis tablet, East Asians used a counting board (*suan zi*) made of bamboo, wood, ivory or bone arranged in columns, Chinese used *suan pan*, Japanese used *soroban* and Russians used *scet*.

Europe and places under their rule replaced the counting-board technology with the pen-and-paper technique based on the Hindu-Arabic numerals. The textbooks that explained their use were known as *algorisms*, which gave the origin to the modern term, *algorithm*. These textbooks were popular and tended to promote the algorism over the abacus.

For example, a famous illustration from the book *Margarita Philosophica* showed the comparison/competition between the abacus and algorism. A female figure represents the Arithmetic who is overseeing two individuals, an older man representing Pythagoras, who is mythically credited for inventing the counting board and a modern man representing Boethius, who is mistakenly credited for inventing the new numerals.

Many believed that the Hindu-Arabic numeral system was superior to the older systems, but it is due to the technological advancement and ease of use. While the Hindu-Arabic numerals were used to record and perform calculations, the Greek and Roman numerals only recorded the results obtained by the counting board. Calculations were performed by fingers or the abacus for these numeral systems.

The positional principle provides the calculation powers of the Hindu-Arabic numbers and the abacus. The place-value principle states that the value of a digit in a number is based on its position relative to other digits (number system) or the value of the counter depends where it is placed on the counter board (abacus). Arithmetic operations on the counting board depends on the counter placement on the board.

ADDITIONAL READING NOTES

TOKENS AND WRITING: THE COGNITIVE DEVELOPMENT

BY DENISE SCHMANDT-BESSERAT

Cognitive archaeology has uncovered some artifacts from the past that document the development of cognitive skills among earlier civilizations. Among these artifacts, they found tokens and tablets that explain the achievements in numeracy and literacy of past generations.

Excavators found mounds on artificial hills that prove the occupation of villagers on the land. Some of the material recovered disclosed the community's reliance on hunting and fishing, while other artifacts disclosed the technological and economical advancement of the community.

The uncovered tokens emphasized the development of cognitive skills among the inhabitants of the land, in particular, they provided an insight on the counting practices in the cultures that created and used the tokens. The tokens also varied in shape and sizes, from cones to spheres to cylinders, all representing various units of something like grain and labour.

THE FUNCTION OF COUNTING IN PREHISTORY

Tokens were found in the Fertile Crescent of the Near East around 7500 BC, which was around the time of the agriculture economy emergence. It is believed that the tokens were used by leaders to keep track of the goods and redistribution depending on the community needs.

THE COGNITIVE SIGNIFICANCE OF TOKENS

Unlike our system of counting, the token system was not universally applicable because it did not use abstract numbers to count a variety of things. In the token system, counting was restricted to the selected units of goods and each category of item having its own counter. This illustrated the one to one correspondence of the token system, each counter counted a specific item. The type of counter (cones, cylinders, spheres) depended on the item (labour, grain, oil).

With the advancement of the redistribution economy, the tokens' shape and design became more complex and intricate to describe a larger variety of items in a growing economy.

THE TRANSITION FROM TOKENS TO WRITING

Tokens were still used until 3300 BC, but there was issues regarding the regulation of the tokens. In order to overcome this issue, envelopes were created to safely keep the tokens within. The envelopes were clay balls, on which the accountants marked them with the token before enclosing them. Due to the innovation surrounding the recording of tokens on envelopes, tokens were getting closer to writing.

TABLETS AND COUNTING

With the formation of city-states, the redistribution economy was continuing to rapidly grow and there became a greater need for written form.

The first form of signs began in 3100 BC with a pointed stylus being used to sketch intricate shapes of the tokens with particular markings.

Secondly, instead of the one to one correspondence (repeating the sign for the item as many times as the number of units to record), another change involved the use of numerals (signs indicating numbers), which was placed before the sign of the token (ex. 1 jar).

Thirdly, the state required writing the recipient or donor of goods' name on the tablet, with new signs — phonograms and using the rebus principle (ex. LU + KA = LUCA).

It was no longer effective to rely on the informality of the hand-crafted containers, so the state administration initiated the standardization of measures. The token and writing systems were a large part of the Near Eastern redistribution economy. Tokens helped the civilization create a written form of the materials which introduced a higher degree of abstraction of information.

TOKENS AND ABSTRACTION

The cognitive significance of the token system was its ability to encourage abstraction. By substituting the actual artifacts with the counters, merchandise was accounted for by abstraction. Some of the benefits of abstraction with tokens include:

- (1) Abstraction of bulk and weight of merchandise
- (2) Abstraction of life or movement
- (3) Abstraction of time
- (4) Simple arithmetic with counters instead of actual items (grains, oil, animals)
- (5) Organizing counters into lines and columns to perform higher level of abstraction and arithmetic
- (6) Organizing goods (symbolized by tokens) according to relative value

After arithmetic operations became common with tokens, the process of the invention of writing was underway.

WRITING AND ABSTRACTION

Abstraction was a complex system that was developed into the evolution of writing. In particular, abstraction was related to the creation of two-dimensional signs, abstract numerals and phonetic signs.

The comparison between tokens and writing displays the complexity of abstraction and its evolution.

- (1) Tokens abstracted information from goods and writing abstracted information from tokens
- (2) Tokens used an one-to-one correspondence, while writing abstracted numbers by using numerals
- (3) Writing created a platform for communication with speech because it represented concrete units of goods (like tokens) *and* immaterial sounds of speech (phonetic syllabic signs for names).

CONCLUSION

During four and half millennia, the token system was initially used to count goods and then went on to abstract information from a more complex set of data. During the course of its development, it is evident that tokens initiated a level of abstraction necessary for future literacy and civilization.

Lecture 2

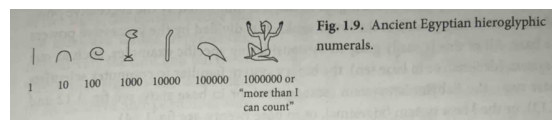
SEPTEMBER 17, 2018

TEXTBOOK NOTES

1.2 WRITING NUMBERS

SYSTEMS OF NUMERALS AND THE POSITIONAL PRINCIPLE

Despite predating writing, tallying evolved into numerals, written signs representing numerical quantities. As one of the earliest forms of writing, numerals consist of 2—40 number signs, grouped based on a number base (usually 5 or 10), combined to form other numbers. Some numerical systems are additive, which means that the value of the number is determined by the sum of the value of all the symbols used to represent it. For example, the Ancient Egyptian system used special signs to signify different values.

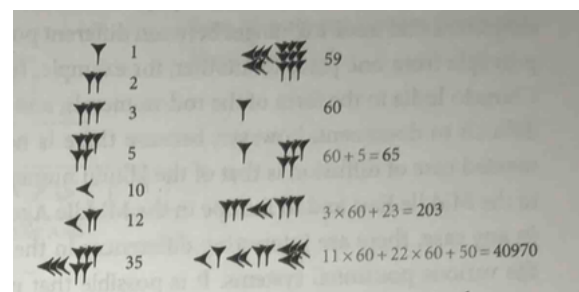


Some systems interpret the values of the numerals as fixed and independent, regardless of position and arrangement of the signs, like the Ancient Egyptian system. Other systems interpret the values of the numerals based on the positional principle, the placement of the digits in the number determine the overall value of the number.

Numerals primarily represent numbers, but can also perform computations (to determine the value of the number). Some cultures have utilized the ability of numerals to compute in their own numeral systems. According to the positional principle (and counting board), the number signs, or digits, do not have a fixed value, but their value is based on their relative positions to other number signs.

Cay tablets, dating late fourth millennium BC, illustrate the earliest use of numerals in Mesopotamia. The evolution of the token imprints into cuneiform (wedge-shaped writing form) serve as a milestone in the process of the invention of writing. The transition also illustrated the interpretation of numerals as independent abstract entities.

The cuneiform system was developed by the Babylonian administrators. Their numerals consisted of two signs, (1) the upright wedge and (2) the sideways wedge to represent unit and ten units respectively. This system was a decimal additive system and a sexagesimal positional system. They did not have a number to represent 0 until after 600 BCE.



Other countries also made use of the positional principle and abacus such as China, India, the Andes region among others. Some believe that trade

facilitated the diffusion of the positional principle among various cultures, but this has not been proven with any evidence.

THE DIFFUSION OF HINDU-ARABIC NUMERALS

While it is unknown when the positional system emerged in India, its existence in the seventh century BCE illustrates its application to the previous numeral system used by the Indians, the Brahmi numerals. The Brahmi numerals consisted of nine Brahmi symbols, but the positional systems supplied a new sign to signify an empty position. The knowledge of the numeral system spread into the Middle East, where several textbooks were written to teach the Indian system of numerals.

Even though the positional principle returned to Mesopotamia, most knowledge pertaining to numerals was lost so Al-Khwarizmi and his contemporaries were unable to connect the information of the new Hindu system to the ancient cuneiform system. From the Latin writings, it became known of the work of Al-Khwarizmi. While the Hindu system was initially used for integer values, it also started to introduce notation for fractional values.

Implementing the Hindu system into the Middle East and Europe was difficult at first due to the contrasting views on the system, especially the need for writing implements. Some schools reverted back to finger counting because it did not require speaking or writing. On the other hand, a mathematician, Abu-al-Hassan (the Euclidean) wrote a book detailing the advantages of the Indian method over finger counting, especially the ease of use, timesaver, precision and efficiency. The mathematician used pen and paper to teach the system, instead of the dust board that was originally used. The ancient numerals traveled into North Africa and Spain.

Fibonacci, son of Italian scholar spread the use of the Hindu-Arabic numeral system to non-academic users. In his famous book, *The Book of Calculation*, he extended the use of the numeral system into commerce and banking by the European society. There were abacus schools that taught practical mathematical knowledge. The growing economy realized the need for more advanced forms of numeracy.

The Hindu system was constantly competing with the older trusted techniques like finger counting and the abacus. Many European countries reverted back to using finger counting to handle transactions because they did not trust the foreign system of record-keeping and transactions. The people also did not appreciate the use of writing implements because the material was not readily available or it was unhygienic (ex. dust boards). Overall, the system was not considered practical

The Hindu system also presented conceptual issues, like the ordering of the numbers. The Arabic read their script backwards, while the Latin read it left to right. The numbers were arranged left to right in decreasing values. This proved to be counterintuitive because the sign value was not determined until the whole number was completed. Early scholars encouraged others to use the positional method in writing their numbers, backwards.

THE PROBLEM OF ZERO

In early history, there was no need for a sign to signify no quantity because the numerals were only used to display quantities. The need arose during the use of a written positional system, as displayed in the Dresden Codex — a Mayan document.

After the conquering of Mesopotamia, The Greeks adopted their methods for fractions and the empty position sign. 'Zero' is a contraction of the Italian word *zefiro*, Arab word *sifr* and Indian word *sunya*. In its early use, it was not considered a numeral but merely a symbol which was kept separate from the other nine digits. The reason being that numerals represented quantities and it would be contradictory to add in a numeral that did not refer to anything but acted as a empty-position marker, it did not qualify as a number.

In order to include the zero into the number system, there was a need to provide greater autonomy to numbers than just reflecting pre-existing objects or quantities. Defining the zero to readers, algorismists had to take extra precaution not to mention that it signified nothing—but that it did "not signify" by itself. Zero could be defined by what it did when it wa properly associated with other symbols.

The Indian astronomer Brahmagupta had already gave an operational definition to zero, and listed the rules fo carrying out arithmetic operations with it:

When zero is added to a number or subtracted from a number, the number remains unchanged; and a number multiplied by zero becomes zero.

LECTURE NOTES

SYSTEMS OF NUMERALS

Numeral are not numbers, they only report numbers based on their notation.

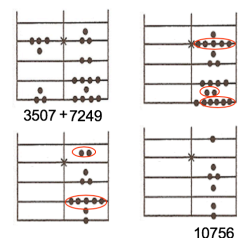
THE ABACUS

Abacus—singular Abaci—plural

An abacus is an object without a base. The first tablet was found in the Greek Islands, around 300 BCE, and named the *Salamis Tablet*. It was heavily used during the Hellenistic period and the Roman Empire. The main use of the abacus was believed to be for money calculations (eg. converting currency). The Darius Vase had illustrations on it that showed the exchequer counting and receiving tributes from people to be given to the King. The tributes were calculated by the abacus in the illustrations, documenting its early use. Even after new numerals were introduced, the abacus still competed with them for a long time. The Romans had a real hand abacus, the secod preserved specimen and is now in the Museo delle Terme in Rome. The Chinese abacus was called *suanpan* and the Japanese abacus was called *soroban*.

THE POSITIONAL PRINCIPLE

In relation to the abacus, the positional principle was a rule for representing numbers and it allowed the formulation of techniques to calculate (addition/ subtraction). The principle dtermined the value of digits.tokens/sign, depending on where it was placed (position of sign determined value).



CLAY COUNTERS/TOKENS

Prehistoric refers to the events that occurred before writing was invented and those events could not be written. Historical events can be documented because of the remains of the culture and writing materials being left behind, like the tokens. Earliest tokens were found in 7000 BCE and the cuneiform writing was introduced in 3000 BCE. Clay objects were common in the past, like tokens, due to material availability of the clay.

Tokens were used for a long time to count quantities of items (ex. livestock, units of good like grain or wine) and for cultural civilizations (to establish method for recording, on a 1:1 basis —1 token = 1 object). Simple tokens were differently shaped to represent differentiated things within a category (ex. royal horses may be represented by cones and labour horses may be represented by spheres). After a while, complex tokens with intricate carvings emerged to represent more complicated counting and larger quantities.

CLAY ENVELOPES

Due to security purposes, temples kept tokens safe inside of a seal that was made of clay, called clay envelopes. If the envelopes were broken, it meant that the tokens were tampered with. In order to overcome this issue, the temples started imprinting on the envelopes to show what was inside. The use of imprints was used for many years because it did not require breaking the envelopes to determine its content. Once they realized that they no longer needed the tokens because they had the imprints to signify the value, the process towards the invention of writing was beginning to emerge.

Over time, envelopes were replaced by tablets, which used the impression of symbols to represent measures of things. Then, the token imprints became numerals, and with numerals there was a higher degree of abstraction than with the other methods. Abstraction referred to representing quantity separate from the object. And then, concrete counting became abstract numbers, which created an inventory of the earliest written document.

SUMMARY

The transition stages involve:

- (1) Tokens were used to represent early methods for counting quantities.
- (2) Over time, the use of imprints on tokens no longer required the tokens
- (3) The system of numerals was developed when the concept of separating quantity from object was introduced by abstract numbers.

—————Break—————

EGYPTIAN NUMERALS

The Egyptians used perishable materials (like papyrus) for writing and due to the rains and over time, the material that documented historical development of their counting system was destroyed. They utilized two forms of writing: (1) Hieroglyphic numerals and (2) Hieratic numerals. Hieroglyphic script was usually found as carvings on stone monuments and for ceremonial purposes, in 3100 BCE. Hieratic script was more symbolic and written on the papyrus (ex. Rhind and Moscow papyrus) for every-day purposes.

Hieroglyphic numerals were a decimal system (base 10), cumulative (accumulate

	1
∩	10
∩	100
∩	1,000
∩	10,000
∩	100,000
∩	1,000,000 or more than I can count

signs) and additive (values added). It was a non-positional system because the position of the digit did not change the value. It was mainly used for royal and complex scriptures.

Hieratic numerals were used for everyday counting, they were a decimal and additive system that used ciphering (coding).



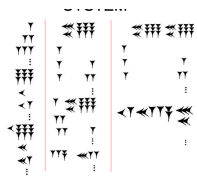
ROMAN NUMERALS

The Romans used a sub-base of value system which is used for monumental and decorative purposes nowadays. They are numerals with sub-base of 5s, cumulative and additive but modified by the subtraction rule. The subtraction rule applies that a sign with a smaller value before a larger value subtracts the larger value because in this system, numerals does not repeat the same sign more than three times.

I	1
V	5
X	10
L	50
C	100
D	500
M	1000

It is a non-positional system because of the subtraction rule, the value does not change based on position.

BABYLONIAN NUMERALS



The Babylonian numerals are sexagesimal (base 60) with sub-base of 10. The symbols would change in rank (downright wedge < upright wedge). It was used for positive integers and fractions. This was a positional system used for large numbers and arithmetic.

The separator sign overcame potential confusion due to empty positions, it was not a numeral.

MAYAN NUMERALS

The Mayan numerals were vigesimal (base 20), except for the third place-value, cumulative with sub-base 5 and possible positional with an empty-position sign. Mayans were located in Central America with a combination of hieroglyphic and ciphered, they emphasized datas and unit of time.

Lecture 3

SEPTEMBER 24, 2018

LECTURE NOTES

THE HINDU-ARABIC SYSTEM

SYSTEMS OF NUMBERALS PART 2

Recap: the *positional principle*, rule for representing numbers in writing or on a counting board, according to which the value that each sign in which it is placed in the number phrase or on the board. Specifically, the positions are assigned, in order, a place-value equal to a power of the base. Each sign or counter will represent one time or, depending on the system, a given number of times the place-value of the position in which it is placed.

The earliest record of Hindu numerals was an inscription found in the Gwarlior temple in 600 CE. Since the inscription was found much after cuneiform writing development in Mesopotamia, it can be hypothesized that the positional principle in other cultures was transmitted to India, where it was modified. But, this is only a hypothesis and cannot be proven.

The Hindu system of numerals was originally the Brahmi system, a non-positional system. It is believed that the positional principle was applied to this ancient system to derive the new signs. In addition to the original signs, there was also the addition to the symbols, a 0 (*sunya*) to signify an empty position. Please note: zero was not a number but a numeral that represented a number. The new Hindu numerals was a decimal (base 10) system, ciphered (the numbers for under base 10 were represented by all different symbols —ex. Hieratic symbols) with an empty-position sign. Note: A system can be ciphered or cumulative (one or the other).

The Hindu numerals were culturally transmitted through the west of India (with historical evidence to prove it). The timeline begins in 600, when the decimal positional system emerged in India. In 700 CE, the system was credited as *the clever method of calculation of the Indians*, by a Syrian scholar. In 800 CE, the Indian diplomatic mission to Baghdad brings astronomical tables with Hindu numerals. Baghdad was the meeting point in many cultures to come together and that is where Islamic astrologers learned the Hindu numerals.

In 900 CE, Persian astrologer Muhammad Ibn Musa al-Khwarizmi wrote *kitab al-hisab al-Hindi* (Book on Indian Calculation). The book was used to transmit culture to European cultures. In 1100, Spain published the first Latin versions of Book on Indian Calculation. In 1000-1100, the Hindu-Arabic numerals appeared in Spain and Italy. The Latin version of al-Khwarizmi's book began with *Dixit Algorismi*. The Hindu-Arabic system became known among mathematical scholars as *algorismus*.

In 1200, Italy, Leonardo of Pisa (aka Fibonacci) learned the Hindu-Arabic numerals and wrote the book *Liber abaci* (*Book of Calculation*), addressed to businessmen. He taught in the schools of abacus. The schools of abacus was a school to teach calculations where the teachers were abacists. They were

commerce schools and during the time the abacus was considered equivalent to calculation so the schools were literally, school of calculations.

In 1300, city of Florence bans Hindu-Arabic figures from bookkeeping. The city feared the widespread of the new numerals as they were quickly growing in commerce (used for double-entry bookkeeping). They suspected that the new numerals were easy to manipulate and considered them a trickery system. During the ancient times, *banning* something meant that it was growing in popularity.

The Islamic continued to develop the system in 1600, including new developments to to develop decimal notation to represent fractions and other numerical quantities like surds.

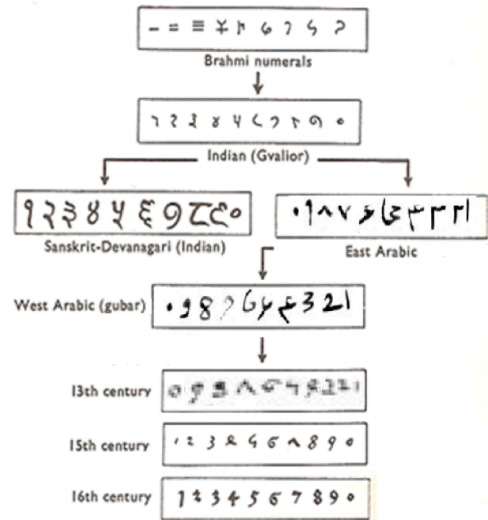
Between 1400-1500, abacists and cossists disseminated the Hindu-Arabic system in Europe as there was lots of resistance against the new numerals. Note: 0 was not considered a numeral, it was kept separate from other numerals as proven by Fibonacci's drawing.

Official scribes avoided using the Indian system because it required equipment. They believed that an appropriate system should not require anything besides the body to secure secrecy and be dignified. The finger-counting system required constant memorization. The new numerals were easier to use, required little precaution and little memorization.

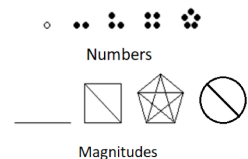
The writing implements (like pen and paper) were necessary for the Indian numerical system therefore the scribes avoided the system or used dust boards as a replacement for the paper and pen. But, the dust boards were considered inelegant and low-class for aristocrats who were the majority of the people learning and using the new system. The signs had no fixed value and numbers appeared to be written backward (ex. 3709, the value of 3 is not known until the whole number is written but the 3 is written first = backwards).

In Greek theoretical mathematics and in the European scholastic tradition, numbers were *multitudes of units*. Numbers were discrete and one was not a number. A sign representing *nothing* was regarded as self-contradictory, philosophically unacceptable. It was counterintuitive that a sign of no value could multiply the value of signs to its left.

The zero does not *signify*, ie. it does not refer to any object, yet it acts upon the digits. It was a 'zero-sign', meaning it was not a symbol of any object, concrete or abstract but a sign about signs.



Numbers and magnitudes in Greek theoretical mathematics



Lecture 4

OCTOBER 01, 2018

TEXTBOOK NOTES

3.1 THE PRIMARY SOURCES OF EGYPTIAN MATHEMATICS

Ancient Egyptian culture has been credited with technological competence in several areas but the sources of their activities have not survived. Greek historian Herodotus credits Egyptians for inventing geometry, from early records of how the civilization divided land. The term 'geometry' derives from *land measurement*.

An Egyptians livelihood depended on the Nile floods, after which *rope-stretchers*, surveyors, had to re-measure the land and set new boundaries. They measured the land using ropes that were marked with knots or paint un unit lengths, the cubit (royal cubit : 52.3 cm = 7palms, small cubit : 45 cm = 6 palms) and the *khet*.

In addition, each farmer owed crop to the state which was calculated by multiplying their land measurements (by surveyor) by the quality of the soil and other conditions. The process involved a team comprising of: stretcher o the cord, holder of cord, tax scribe, clerk of the land, and other administrative officials.

Besides architecture and land survey, mathematical competence was required in several other areas of Egyptian life like labour, managing resources, pharmacy, metallurgy among others. For example, the Pyramid of Giza required knowledge of several mathematical concepts in its construction.

Egyptians were knowledgeable in mathematical studies due to the developments in their time; however, very few written documents have survived. The main reason for the low survival rate is due to the perishable nature of their writing instruments —papyrus, wood, and pottery sherds. Two surviving documents include the Rhind Papyrus (1650 BCE) and Moscow Papyrus (1850 BCE).

3.2 THE MATHEMATICAL POPYRI

The two documents contain mathematical exercises in hieratic script which gives the impression that they were textbooks for students and teachers.

The Rhind Papyrus is a mathematical document with two division tables and 84 problems and from the document it was interpreted that the Egyptians acknowledge fractions as 'part' of

whole quantities. The first group of problems were the *aha* problems: where a numerical quantity was required under a given condition. These type of problems provided the skills to deal with word problems about real-life situations. The exercises were particular problems that were meant to exemplify a general method. The solutions were provided as algorithms, a procedure with a series of instructions to be followed step by step.

3.3 SCRIBAL TRAINING, PROFESSIONALISM, AND THE PRACTICAL CHARACTER OF EGYPTIAN MATH

Since Egypt had a group of professionally practicing mathematicians, it can be inferred that the civilization valued it enough to provide support. The mathematical knowledge that the state emphasized was practical math that could be useful.

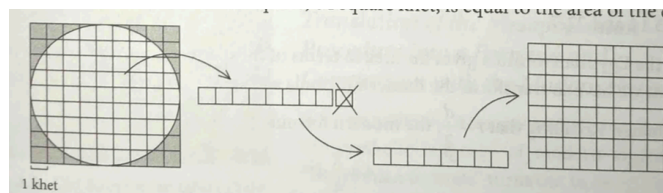
There were some problems in the papyrus that introduced abstract mathematical concepts not applicable to real-life situations (ex. large circle of land?). It is believed that these methods helped scribes expand their knowledge and skills attached with the exercises and concepts.

3.4 MATHEMATICAL FORM AND CONTENT, AND THE PROBLEM OF 'TRANSLATIONS'

Traditional Egyptian problems are better understood when they are translated into modern algebraic problems like *rhetorical algebra*, a primitive and verbal form of algebra that is able to solve linear equations. The whole point is that algebra is universal and timeless so it should be able to translate and adjust to changing times. 'Translations' are historians that consider the social and cultural context of mathematical content.

3.5 EARLY CALCULATIONS OF THE AREA OF A CIRCLE AND SOME WARNINGS ON THE 'HISTORY OF PI'

Problem 50 of the Rhind Papyrus deals with a circle. It can be translated into a geometrical procedure by converting the circle into a square:

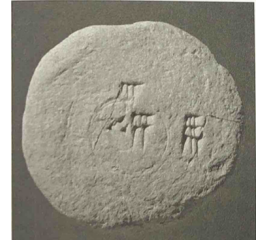


The Egyptians used $1/9$ as a constant in their formula. If the Egyptian formula was converted into the modern formula for the area of a circle, it would imitate it, except that their value of pi was a little off (0.6%): $A = (\pi/4) d^2$

MEANWHILE IN MESOPOTAMIA

THE BABYLONIAN CALCULATION FOR THE AREA OF A CIRCLE

One of the clay tablets discovered from ancient Mesopotamia contain geographical sketches accompanied by numbers. A particular tablet involves determining the area of a circle:



The Babylonians algorithm used to calculate the area of the circle consisted of squaring the circumference and then multiplying the result by the fixed coefficient, $1/12$.

If the Mesopotamian procedure was translated into a modern formula and compared, it would be $A = (1/12) C^2$ and the formula for c , using $r = c/2\pi$. The Babylonian approximation for pi was 3, 4% off the modern value of it.

DISCUSSION

Instead of converting the ancient formulas into modern ones, it is important to analyze the historical relevance and context in which the formulas were utilized. The Babylonian method for calculating the area of a circle differs from ours, but some similarities can be drawn. Historians conclude that Babylonians defined figures by their boundary lines (perimeters and circumferences) while we focus on radius (middle). The ancient Egyptians and Babylonians were not interested in determining exact formulas, but with effective practical calculations.

ADDITIONAL READING NOTES (LECTURE NOTES)

EARLY MATHEMATICS IN MESOPOTAMIA

Ancient Mesopotamia developed over the fourth millennium BCE into a complex society consisting of agriculture and trade practices. Over time, work became specialized with a clear division of classes and socio-economic life was governed by the temple. The separate city-states were taken over by kingdoms and new governments over the course of a long political history.

There were old written documents left behind which detailed the growth of the ancient mathematical culture. The scribes were made from clay and once it was dried, it became very durable. After the scribal tradition disappeared, it became difficult to track the progress of Mesopotamian mathematical development. Since the oldest scribes were dated to 1800-1600 BCE, they are often credited as Babylonian mathematics because that was the period Babylon came into power.

Ancient scribes showed an extended use of the Pythagorean formula, linear and quadratic equations and an overall cuneiform positional sexagesimal system. Since the initial tablets were stolen, their original location is unknown but the context was translated into modern mathematical symbolism. Translation in mathematics was guided by the fact that the concepts are universal and independent of time and culture.

Mesopotamians dealt with accounting by means of clay counters or tokens, before urbanization and the invention of writing. The tokens represented quantity and types of objects on an one-to-one correspondence. Over time the concept of writing emerged when impressions of the tokens with their quantity symbols were kept on a clay surface. Finally, the impressions overtook the need for the tokens and they represented numerical quantities.

The temple or administrative taught students about the numerical method in school because the tablets belonged to the schooling and professional life of the scribes. There is also evidence that there was an interest in exploring the mathematical concepts beyond the real-life problems, *supra-utilitarian*. This concept was considered *mathematics for its own sake* which gave rise to pure/real mathematics (study of mathematical entities and relations without concern for their connections to the real world).

Abstract quantities mark the initial divergence of mathematics from the practical concept into a discipline in its own right. Standardized token-shaped numerals was a major step in the process of abstraction; however, this movement was slow and patchy. The job of scribes involved *metrology*, the knowledge and use of systems of units of measurement. The government imposed metrological reforms to increase uniformity and standardization and the systems were expanded to include more multiples and subunits.

The place-value system was used to operate on the written sexagesimal numerals. Due to the sexagesimal system, new tablets were introduced to perform complex arithmetic with multiplication, reciprocals, numeral constants and parameters.

The Old Babylonian period had the largest group of cuneiform tablets which can be classified into numerical tables and school problems and exercises. The tablets show the progression of mathematical concepts into abstract techniques, beyond the need of the scribal work (real life usage). Many problems that scribes dealt with were geometrical, the Old Babylonian tablets show shapes and line drawings around the second millennium.

The Old Babylonian tablets show a connection between numerical and geometrical problems, in the absence of a visual diagram (which could have just perished over time). Mesopotamian scribes understood geometrical figures as defined primarily by their boundaries like square shape is associated by its sides or circle area is associated by the circumference. A famous example includes the round tablet part of the Yale Babylonian collection, YBC 7302, showing a circle with three



markings (probably the sexagesimal numerals) that can be identified as calculations of the area of a circle. Other tablets also show the knowledge of the relationship between the side and diagonal side of a square, that was codified by Greek geometers as *Pythagoras's theorem*.

The early geometrical problems were regarding the method of the completion of the square, which are equivalent to the quadratic equations of modern algebra. It was also believed that early Babylonian work was most algebra, but this is debatable.

Comparing the Babylonian methods to modern counterparts brought the shortcomings of the Babylonian methods forward: symbolism for operations were imprecise and redundant which prevented developing theories for the equations (ex. quadratic formula) because the common structure of similar equations was concealed by the vague symbolism and the purpose of the method was to be memorized as oppose to understood. But since the teachings were mostly oral, it can be inferred that education was limited due to its social environment.

The Babylonian applied their mathematical concepts to real-life situations in order to understand them. Simple operations like adding, subtracting, dividing and multiplying were essentially *joining two figures together, cutting out/tear off* and the cut-and-paste method for area calculations (ex. completion of the square).

Abstract numbers are a standard representation of all kinds of quantities in modern mathematics, but for Old Babylonian scribes, the standard representation was that of imaginary rectangles and their sides.

Lecture 5

OCTOBER 15, 2018

LECTURE NOTES

GREEK MATHEMATICS

Greek mathematics was demonstrative (instead of algorithmic like the Egyptians), it was based on the concept of proving truth of certain general statements about abstract things. Unlike Egyptians, there was no special institution that taught and emphasized the teachings of numerals.

Greece was peninsula, surrounded by mountains and around water, it did not favour centralized states like Mesopotamia. The Hellenic period was considered the classic Greek period as there was growth of arts and cultures,, city-states and saw the rise of mathematics and philosophy originate. However, it is unknown which one came first and this continues to be a debate today. The 500s, saw the emergence of solid evidence of mathematics due to Hippocrates, who wrote a book. The 600s became the Hellenistic period that saw the rise in potentation of cultural exchange. Famous individuals in the 6th century include Eudoxus, Archytas, Archimedes, Euclid, Apollonius and Hipparchus. The Greek were intrigued by the Mesopotamian body of data because of their interest in astronomy. The common era became after the 1st century.

GREEK PRACTICAL MATHEMATICS

There was no written source of practical mathematics and so archaeologists have to determine the proof this form of mathematics through other sources. For example, there was evidence that the colonies in Greece were allotted to veterans, which required systematic allocation by measuring and dividing the land equally. It is believed that in order to allocate the land efficiently, there would be a need to know some geometry. Another example includes the early engineers and the success of their mission to dig a tunnel under the mountains from two sides and meet in the middle. This showed their knowledge of mathematics

Since the Greek economy was created after the invention of money, they made use of money. For example, tributes required by the state were paid by money, not goods like in Mesopotamia. There is indirect evience from monuments of city-states that prove the recording and accounting of the money exchange as carved on the monuments. This also proved that the people wanted to know how the money was spent and kept so they could use the carvings on the monuments to check on the balance of accounts. The Greek were known as free-traders.

GREEK THEORETICAL MATHEMATICS

Theoretical mathematics was communicated through books and is sometimes termed as philosophical mathematics because of how its origin coincided with philosophy origins. There are book fragementes remaining that prove the existence of theoretical mathematics among the Greek; however, there is no primary source because it was all destroyed. The only remaining fragment belongs to a book by Euclid and even that is not considered a primary manuscript. It is considered a transmission of his original text that may include addtions and errors that would not be found in the original.

Reproduction of ancient books was difficult to do due to the scarcity of parshment paper and would only be done for very important ones. The main reason as to why we lost the ancient Greek books was because no one thought or cared to reproduce the earlier ones that led up to Euclid's discovery, except for Euclid's book and those after him which were also not an original reproduction.

Archimedes is recognized as the greatest mathematician. The transmitted texts of some of his books can be found, a particular transmission was done on a prayer book. The book of prayer was found by a monk in Istanbul and a theologist noticed something below the prayers in the book. Underneath the prayers, there were geometrical

statements and the theologist determined that the prayer book was written on a book by Archimedes which was reused for prayer. Since parchment was scarce and expensive, reusing the paper was a common practice and in the case of the prayer book, the original monk scraped the ink.

In general, the material sources of practical Greek mathematics was derived from indirect evidence found in buildings and published pictures. The primary sources of practical mathematics was found from the tunnels dug and the temples. Book sources of theoretical mathematics only exist **after** Euclid. Overall, less is known about Greek theoretical mathematics than Babylonian mathematics.

Western mathematics is commonly viewed as originating from Greek theoretical mathematics. 'Our evidence concerning pre-Euclidean mathematics is so indirect and fragmentary that the greatest part of the stories we are now told about period must be inventions.' - D. Fowler, *The mathematics of Plato's Academy*, 1999. Pre-Euclidean mathematics existed before 300s, which is considered the beginning of theoretical mathematics, while the Post-Euclidean mathematics was after 300s.

The Greeks developed a unique kind of mathematics, aimed to produce absolute truths about abstract objects and centered on the requirement of rigorous formal proof. The earliest reliable evidence of Greek theoretical mathematics is from the second half of the fifth century BCE. We do not know exactly how and when it began. It emerged in the context of the life of the polis, and the habit of debate.

LEGENDS ABOUT THE ORIGINS OF THEORETICAL MATHEMATICS

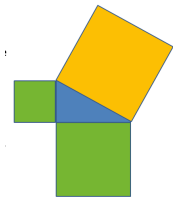
The image named *School of Athens* by Raphael includes early images of Pythagoras and other mathematicians. Thales (ca. 620-550 BCE), who is considered the father of theoretical mathematics, proved some basic theorems on triangles. Pythagoras of Samos is credited with the discovery of harmonic ratios and the belief that 'everything is number' and with discovering the Pythagorean theorem.

WHO WAS PYTHAGORAS?

Pythagoras (ca. 570-490 BCE) was the founder of a religious-political sect, the Pythagoreans in Croton, Magna Graecia, in the 6th century BCE. There is not historical evidence that he practiced mathematics at all. Reportedly, he and his followers attributed mystical meanings to certain numbers and shapes. Later Pythagoreans and other philosophers used mathematical ideas to explain certain phenomena in the physical world, and attributed them to Pythagoras himself. E.g musical harmonies and astronomical cycles as numerical ratios.

The legend of 'Pythagoras the father of mathematics' grew over time, and many apocryphal tales of his alleged mathematical exploits were invented, e.g, that he discovered the theorem that is today named after him. The myth of Pythagoras as the father of mathematics is still firmly lodged in mathematical folklore, although it has no historical basis. It has emotional value as an allegory of the origins of Greek theoretical mathematics.

The square on the diagonal of a rectangle is equal to the sum of the squares on the sides of the rectangle, which is today known as *Pythagoras Theorem*. A case of 'Stigler's Law of Eponymy': There is no scientific discovery named after the original discoverer.



THE PYTHAGOREAN RULE

The Pythagorean rule in old Babylonian mathematics involved the block-puzzle technique, *zhou bi suan jing* and Euclid's elements.

The Block-Puzzle technique aka cut-and-paste technique, involves composing, decomposing, and re-arranging geometrical figures as if they were pieces of a block puzzle. The simplest and easiest case involves the square which extends the techniques to rectangles. The *Pythagorean* rule in the *zhou bi suan jing* (1046-256 BCE) emerged from Zhao Shuang's commentary [Arithmetical classic of the Gnomon and the Circular paths of the heaven]. The *Pythagorean* theorem in Euclid's elements includes the oldest written version of the theorem today known as Pythagoras theorem. Euclid did not name his theorem after Pythagoras, nor did he make any mention of

Pythagoras. The theorem: *Elements*, Proposition I, 47: "In right-angled triangles the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle."

TEXTBOOK NOTES - PG 59-80

CHAPTER 4 – GREEK MATHEMATICS

4-1 THE SOURCES OF GREEK MATHEMATICS

What we know about Greek mathematics is transmitted through books—texts and their accompanying folklore. Famous people include Pythagoras, Euclid, and Archimedes. However, very few of the original documents have survived while the great library of Alexandria, library in Pergamon and public and private libraries in Imperial Rome were destroyed in second and fifth centuries of our era. There has been efforts to interpret and imitate the Greek mathematics, but the documents that remain today are not primary sources, they represent copies of copies, translation, secondary accounts and commentaries created much after their original author's demise.

4-2 PRACTICAL AND THEORETICAL MATHEMATICS IN ANCIENT GREECE

Early Greek culture experienced three major periods:

- (1) The Archaic period (8-6th century BCE)
- (2) The Classical period (6-4th century BCE)
- (3) Hellenistic period (3-4th century BCE)

The Hellenistic period is known as the Golden Age of Greek mathematics because it saw the largest mathematical development of the previous two centuries. It is still unknown how much of Greek mathematics is derived from other civilizations and the relationship between mathematics and philosophy.

Remains in Ancient Greek attest to a high degree of knowledge pertaining to numeracy and geometrical competence. For example, their cities were built on grids that required knowledge of geometrical patterns and straight lines. An example of engineering existed by Eupalinos who used extremely accurate measurements and triangulations to construct a tunnel underneath a mountain.

While it is known that Greek society had professionals who practiced mathematical exercises, it is unknown how they came to be. Greek culture did not provide exclusivity to those with knowledge of numeracy and literacy. Common citizens were required to have knowledge on mathematics. The Greek made use of money, so mathematics was used in as a means of accountability (economic purposes) as well as for social purposes.

The mercantile economy became wealthy, while the upper class focused on leisure activities. During this time, philosophy emerged into society, but around the same time as the development of theoretical mathematics. Theoretical mathematics focused on determining the general truths of abstract objects, it was demonstrative. Modern research on ancient Greek mathematics focuses on theoretical mathematics, while practical mathematics became invisible.

Around the fifth century BCE, mathematics became more specialized into four *mathemata*: arithmetic, geometry, astronomy, and harmonics. Famous philosopher Plato portrayed that Socrates was a supporter of abstraction, while historian Xenophon portrayed him as dismissive of theoretical mathematics.

4-3 THE LIFE OF THE POLIS, THE HABIT OF DEBATE, AND THE REQUIREMENT OF PROOF

While the Greek were not the first to develop mathematical techniques, they were the first to raise meta-mathematical concerns regarding the nature of mathematical knowledge and methodology of mathematical research. Greek mathematics introduced the notion of mathematical proof as a requirement into order for an idea to be central to mathematical theory.

The Greek culture consisted of a leisure class that focused on intellectual pursuit and literacy, a mercantile class who required knowledge of numeracy and the emergence of commerce and travel enhanced their culture. The Greek culture during the pre-classic and classical period experienced many small-city states, *poleis* that have a variety of forms of government. In particular, democratic societies utilized the use of public assemblies that required people to participate in the argumentation because the culture thrived on transparency and accountability.

Mathematics stemmed from the same cultural environment as the art of rhetoric which aided in the insistence of logical proof of concepts. The source of the proof requirement is unknown because there is not surviving fragment of a written source of the early period. The two mathematicians that are often credited for beginning mathematics were Thales of Miletus and Pythagoras of Samos, but these are not true tales because there is no evidence to back them up.

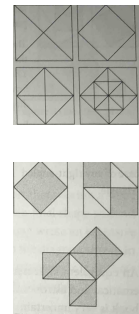
The accreditation of proving mathematics is known as a foundational myth. Foundational myths may not be completely inaccurate as they do contain some historical truthfulness, but they are created for didactical purposes to encourage further research. Thales is credited as a man of great wisdom, but there is no proof regarding his alleged successes like proving theorems. Pythagoras has also made achievements that are not proven by support, especially in regards to the theorem in his namesake. Early proof of the Pythagorean theorem was found in Euclid's *Elements*.

Greek philosophers are divided into two groups: pre-Socratics —patchy historical evidence of mathematical proofs and Socratics —clear evidence of activity in theoretical mathematics surfaced during this time.

4-4 PYTHAGORAS AND THE SO-CALLED PYTHAGOREAN THEOREM: BUSTING THE MYTH

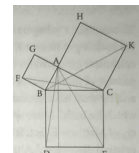
Pythagoras was a political and religious leader that traveled from the island of Samos to the Greek colony of Croton in Magna Graecia. He formed a sect/cult, the Pythagorean brotherhood, that engaged in political violence. The school started as providing rules of conduct but over time it formed a strong mathematical liking. The Pythagoreans were interested in numbers, but they did not consider 1 a number, but as a origin of numbers. Pythagoras was considered a mathematician after Plato fabrication.

There is proof that the relationship between the sides of a right-angled triangle was known before Pythagoras, during the Babylonian period in the Middle East. Ancient geometers discovered the *block puzzle* or *cut-and-paste* technique that decomposes the early tile patterns found on clay tablets into triangular pieces, and rearranging them to form a new figure. From this, they obtained the demonstrations that the square on the diagonal is equal to the sum of the squares on the sides. References to the side-diagonal relation are found among the Babylonian, in the Indian religious text, *Veda*, and in ancient Chinese mathematical text, *the Gnomon of the Zhou*.



The Pythagorean theorem was compiled by the works of many different people whose basis and validity of algorithms were made evident by the means of the block puzzle manipulations. While Pythagoras may have been a contributor to the theorem, there is no evidence that he was interested in rectangles, diagonals, or right-handed triangles.

The theorem was transformed from a practical rule by Euclid in this book, *Elements*. He proved it using a lettering diagram. This diagram had carefully labelled elements that bring attention to the diagram as a static set, unlike the earlier diagrams that were transformative (showed many types to show the transformations from step one to step three).



Pythagoras is a classical impersonation of the Matthew Effect, where scientific celebrities get credit for the results of more obscure fellows. Pythagoras has gained more momentum over time where people like to associate the theorem with him as well, especially in schools.

Lecture 6

OCTOBER 22, 2018

LECTURE NOTES

GREEK MATHEMATICS PART 2

A new kind of mathematics emerged in Greece around the mid-5th century BCE, it was theoretical and self-critical and part of the liberal education of the Greek citizen, alongside with or as part of philosophy. Mathematical proof is a logical argument constructed to show explicitly the validity of each proposition. Its historical context lies in participation in the political life of the *polis*, and habit of debate.

Plato introduced his Theory of Forms, which describes sensible objects and forms (or ideas). Sensible objects are changing, particular, known through sense experience and objects of uncertain knowledge. On the other hand, forms are changeless and eternal, universal, known through reason and objects of certain knowledge.

According to Plato, it would be appropriate to those who share the highest offices in the city to turn to calculation and take it up, not as laymen do, but to study the nature of numbers by means of understanding itself, not like tradesmen and retailers, for the sake of buying and selling, but for the sake of war and for ease in turning the soul upward, away from becoming and toward truth and being. (380 BCE).

The earliest surviving text containing an extended discussion of theoretical mathematics involved a mathematical problem, discussed by Plato in the Socratic dialogue, *Meno* (ca. 380 BCE). In a philosophical discussion with Meno, Socrates (Plato's character) calls a slave boy and asks him, "given a square of side 2ft, what is the side of a square twice its size?" Socrates, drawing a diagram and steering the boy with questions shows that the boy's initial answer (4ft) is wrong and guides him to the correct answer: the side of the double square is equal to the diagonal of the original square.

The Babylonian solution to doubling the square was algorithmic and numerical. They calculated successive approximations of a number that when multiplied by itself is close enough to 2 (e.g. 1.41 or 1.414, or 1.4142, or 1.41422, etc.). They multiplied the length of the side of the square by this number to find the approximate length of the diagonal. Also, they multiplied the length of the diagonal by itself and checked if it was close enough to twice the area of the original square and repeating it until find a good-enough approximation.

On the other hand, the Greek solution was demonstrative and geometrical. They drew the diagonal of the square, constructed the square that has the diagonal as its side (ie. the 'square on the diagonal') and geometrically displayed the constructed square as is exactly twice as large as the initial square.

The quantities in Greek theoretical mathematics was numbers and magnitudes. Numbers, *arithmoi*, were known as multitude of units, the unit is indivisible and is the element of which numbers are composed, but is not itself a number. Numbers are also discrete and *in succession*: between a number and the next there is no other number. Magnitudes are continuous quantities, such as lines, surfaces or solid figures. The parts of a magnitude touch one another and the touching limits become one and the same. A continuous quantity cannot be composed of discrete elements, for example, a line cannot be made of points because points have no parts, between two points on a line there always are other points and there is no such thing as an actual infinity of points to make up a line since *infinity is only potential*.

Mathemata, originally meant things to be learned and *mathematikos* is a learned person. From mid-fifth century BCE, the four *mathemata* are arithmetic, geometry, astronomy, and harmonics. The topics were used differently by Greek authors, sometimes referring to theoretical topics only, and sometimes to a mix of practical and theoretical. These four topics were used in the curriculum of philosophical schools and later adopted by early universities as the *Quadrivium*.

A Greek **ratio** (*logos*) is a relation between quantities of the same kind. It is not a number or quantity, it exists only between homogeneous quantities (ie. quantities that can be compared to one another —equal, larger, smaller). For example, a ratio of lengths, a ratio of areas, and a ratio of numbers. Exact quantitative evaluation of a ratio: how many times the larger contains the smaller. A **proportion** is an equality of ratios and the ratio of two magnitude can be equal to the ratio of two numbers. For instance, *q1 is to q2 as 3 is to 2* means q1 contains q2 as many times as 3 contains 2, 2 times q1 is equal to 3 times q2 and there is a u (homogeneous with q1 and q2) such that $q1=3u$ and $q2=2u$.

How many times is the smaller contained in the larger? For example, the diagonal and side of a square, diagonal and side of a regular pentagon, circumference and diameter of a circle. The ratio is independent from size of the figure. The essential nature of a figure lies in the ratios between its homogeneous, not in its size. Also, ratio (*logos*) means shape, inner order, beauty, reason and meaning.

Hippocrates of Chios wrote the first textbook of theoretical mathematics, titled *The Elements*. He tackled the problem of 'squaring the circle': what is the exact area of a circle? In other words, what is the side of the square that has an area *exactly equal* to that of a given circle? And how can we know it with *certainty*? Hippocrates could not solve the problem, but he *proved* that circles are to one another as the squares of their diameters. The problem of squaring the circle in the Rhind papyrus was that the ancient scribes knew how to square the circle *approximately*.

Hippocrates of Chios also tackled the problem of 'doubling the cube': how to construct a cube twice as large as a given cube. He proved that the side of the double cube is in a continued proportion with the side of the given cube. Archytas of Tarentum found an exact solution that involved moving figures: a *mechanical* solution.

The critical analysis of measurement believes practical measurement is always uncertain and approximate. The goal of practical mathematics is to find numbers that are good enough for practical purposes. The goals of theoretical mathematics are exactitude and certainty. Given two homogeneous quantities, the smaller measures the larger if it contained in the larger an exact number of times. A quantity is a *common* measure of other homogeneous quantities if measures all of them. When two quantities have a common measure, their *ration can be expressed* exactly as the ratio of their numerical measures.

The discovery of incommensurability believes that practical mathematicians operated *as if* homogeneous quantities always had a common measure. But when the Greeks analyzed measurement critically, sometime around 450 BCE, they discovered that this is not always true. Magnitudes of the same kind do not always have a common measure! They called magnitudes having a common measure *commensurable*, and magnitudes having no common measure *incommensurable*.

The Greek method of calculating ratios asked how many times does the long side of this rectangle contain the long side? (once with a remainder) How many times does the short side contain the remainder? (three times exactly) The ratio of short to long side is 'once and thrice'. (it is equal to the ratio of 3 and 4, the short side contains the remainder 3 times, the long side 4 times). How many times does the first remainder contain the smaller remainder? (twice) The ratio of short side to diagonal is 'once, once, and twice'. (it is equal to the ratio of 3 and 5, the short side contains the smaller remainder 3 times, the diagonal 5 times). The last remainder is the common measure and the method is called 'alternate difference', 'mutual subtraction', 'Euclid's algorithm', or 'anthyphalresis'.

Let's try to find the ratio of side and diagonal of a square using the method of alternate differences. The diagonal contains the side once with a remainder, r1. How many times does the side contain the remainder? To answer, we must find how many times r1 is contained in the diagonal of its own square, d1. The question returns endlessly to itself at ever smaller scale and the side-diagonal ratio is self-recursive.

Similar cases of endless self-recurring ratios occur in other regular polygons. For example, when trying to express numerically the ratio of side and diagonal of a regular pentagon. In this case too, it turns out to be impossible to find a common measure between side and diagonal. Hence, there is no numerical ratio that is exactly equal to the side-diagonal ratio.

Those magnitudes are said to be commensurable which are measured by the same measure, and those incommensurable which cannot have any common measure (Euclid's Elements, X Def.1). Straight lines are commensurable in square when the squares on them are measured by the same area, and incommensurable in square when the squares on them cannot possibly have any area as a common measure (Euclid's Elements, X Def.2).

With these hypotheses, it is proved that there exist straight lines infinite in multitude which are commensurable and incommensurable respectively, some in length only, and others in square also, with an assigned straight line. Let then the assigned straight line be called rational, and those straight lines which are commensurable with it, whether in length or in square, or in square only, rational, but those that are incommensurable with it irrational (Euclid's Elements, X Def.3).

TEXTBOOK NOTES

CHAPTER 4: GREEK MATHEMATICS (continued)

4-5 THE DISCOVERY OF INCOMMENSURABILITY

The Pythagoreans discovered that the side and diagonal of a square are incommensurable; they cannot be measured with a similar unit of measurement. Some credit this discovery to Hippasus of Metapontum who may have been drowned at sea for divulging the law of secrecy or for refuting the dogma that everything is number.

Two lines that are measured by a common unit can be represented as a ratio of whole numbers. The largest common unit can also be expressed as the minimum number ratio (for example, 6:4 becomes 3:2). The Greeks referred to two incommensurable quantities as unaccountable or inexpressible, nowadays known as irrational. In mathematical terms, an irrational quantity is a quantity that is incommensurable with the unit.

Some credit the rise of incommensurability to the discovery of *irrationality*. For example, Plato used the mathematical discovery of irrationality to achieve a philosophical thinking of it. Due to his interest in mathematics, Plato founded the school in Athens, *The Academy*. His students further researched irrationalities to discover more quantities besides the ratio of side and diagonal of a square.

The Greeks classified quantities into two classes:

- (1) Discrete Quantities or 'Numbers'
- (2) Continuous Quantities or 'Magnitudes'
—subdivided into rational and irrational quantities

While we classify numbers into inclusive sets: natural, integer, rational, irrational, real and more, a major part of our knowledge on them derives from the ancient Pythagorean discovery of irrational quantities.

Aristotle provided a hint to the origin of incommensurability through his writings. He mentioned that a proposition can be proved by proving that it is absurd or contradictory to deny the consequences of a proposition. This alluded to the more general proof provided in Euclid's Book X of *The Elements*. In the case of the Pythagorean formula, the proposition to prove that side and diagonal of a square are incommensurable, its negation (side and diagonal have a common measure) would have to be proven as an impossible conclusion.

The discovery of incommensurability led to a major development and departure of Greek theoretical mathematics from practical mathematics. The Greeks believed that irrational magnitudes (ex. square root 2) could not be represented as numbers which led them to separate theoretical geometry from arithmetic. They produced figures, lines, angles, polygons and more by referring to the expression of their lengths and areas as irrational quantities (cannot be measured by a similar unit of measurement). Also, magnitudes were expressed similar to ratios.

Before the Greeks' discovery of incommensurability, the Old Babylonian scribes calculated the length of the diagonal of a square, as drawn on a famous tablet from a scribal school. The diagonal of the square was calculated by multiplying the side by a coefficient (from a coefficient table). Their results were very close approximations of the

Greek theorists but not exact quantities because irrational quantities cannot be expressed exactly as ratio of whole numbers or as a finite sequence of digits in a positional system of numerals.

4-6 HIPPOCRATES OF CHIOS, THE FIRST ELEMENTS OF GEOMETRY, AND THE SQUARING OF THE CIRCLE

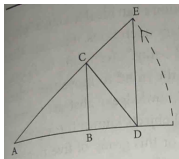
There were two mathematicians, Hippocrates of Chios and Archytas of Tarentum, who worked on two of three famous classical problems of Greek mathematics:

(1) The Squaring of the Circle

Hippocrates of Chios proved that circles are related to each other similarly as squares are their diameters, which was a crucial development in finding the area of the circle. While he did not arrive at the solution, he was able to find the areas of circular figures shaped like crescent moons. He also wrote literature on geometrical theorems to provide order to the unsystematic development of geometry.

4-7 ARCHYTAS OF TARENTUM AND THE DUPLICATION OF THE CUBE

(2) The Duplication of the Cube



The duplication of the cube originated from the island of Delos, where citizens were suffering from the plague and were required to double the size of an altar to Apollo (a cubic shape) to stop the epidemics. Hippocrates reduced the problem to the search for two means proportional between the side of the given cube and its double.

Archytas provided his solution to duplicating the cube consisting of two rotating figures, a triangle and a semi-circle, and their intersection with a fixed semi-cylinder.

The solutions provided by both mathematicians did not solve the problems and they were considered too mechanical by Plato. The methods used employed Euclid's definitions, axioms and theorems in a *straightedge and compass method* to provide a restrictive solution to the two problems.

4-8 PLATO AND MATHEMATICS

While Plato was not a mathematician, he promoted it and was influenced by it. He centered his belief on the notion that there is an ideal world of *ideas and forms* that exist beyond the imperfect and unstable world of sense experiences. According to Plato, mathematics trained the mind to move from experiencing sensible objects to the knowledge of abstract forms by studying mathematics. Plato also used mathematical knowledge in his philosophical dialogue to prove his propositions. Plato's mathematical examples presented as numerical problems, but were solved by geometrical expression.

Lecture 7

OCTOBER 29, 2018

LECTURE NOTES

EUCLID AND THE ELEMENTS

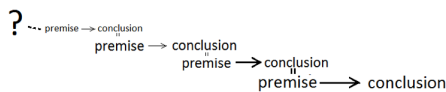
Three definitions for the way of knowing:

- (1) *Techne* (Lat. *ars*—>art): know-how, practical expertise
- (2) *Doxa* (Lat. *opinio*—>opinion): probable knowledge
- (3) *Episteme* (Lat. *scientia*—>science): certain knowledge, based on demonstration or proof

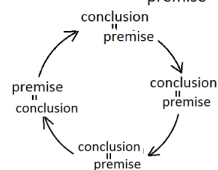
Inferential reasoning involves deduction and induction. Inference means to reason from a premise to a conclusion (ex. If A then B —symbolically $A \rightarrow B$). Deduction means if A then certainly B, for example, A is all men are mortal and Socrates is a man and B is Socrates is mortal. Induction means if A then probably B, for example, A represents the average lifespan of a man is 80 years and very few men live past 100, Socrates is a man and he is 99 and B is Socrates will probably live past 100.

How can knowledge every be certain?

- Infinite regress

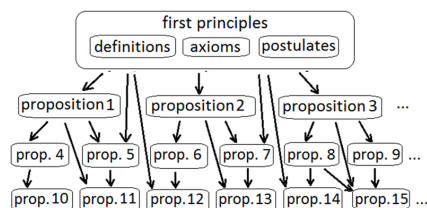


- Circularity



Scientific reasoning according to Aristotle involves propositions, first principles and scientific demonstration or proof. Propositions are demonstrable truths. First principles are truths accepted without demonstration and include definitions, axioms (or common notions) and postulates (or special notions). Scientific demonstration or proof is an explicit argument that shows how a proposition follows with logical necessity from first principles or previously demonstrated propositions.

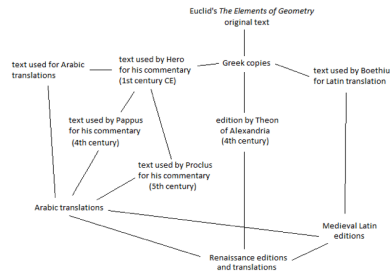
The axiomatic-deductive system:



Elements was composed by Euclid around 300 BCE and it was the oldest Greek treatise of mathematics preserved in its entirety. It was considered a comprehensive compilation of Greek mathematics and was organized by Euclid into an axiomatic deductive system. It included first principles (definitions, common notions-or axioms, and postulates), a tightly organized chain of proven propositions (theorems and problems) and proofs (logical arguments showing

explicitly the deductive inferences that led to every proposition. The book dominated mathematical education for 2200 years and became the ideal model of a scientific treatise.

The oldest surviving treatise of theoretical mathematics of the Euclid's *Elements* was found in Oxyrhynchus, Egypt in 1896 on a papyrus. The oldest extant copy was found in Constantinople in 888 CE. The *Elements* before printing:



Euclid's *Elements* was organized using an axiomatic-deductive system as a deductive system of propositions to avoid circularity and infinite regress, he started with a set of definitions and first principles, axioms and postulates. Every other proposition in the system is a **theorem**, proven from the first principles directly or through previously proven demonstrated theorems. It includes 13 'books' (ie chapters or scrolls) with 23 starting definitions, and a number of other definitions added as needed in the following books, 5 postulates and 5 common notions and 465 propositions and their proofs.

The content and composition of Euclid's *Elements* included several Books (considered chapters):

Books I to VI: Plane Geometry

I: Starting definitions, postulates, common notions, triangles, parallel lines, Pythagoras theorem

II: Transformation of areas

III and IV: Circles and regular polygons

V and VI: Theory and regular polygons

Books VII to IX: Theory of Numbers

Book X: Theory of irrational magnitudes

Books XI to XIII: Solid Geometry

Euclid's definitions include:

- (1) A point is that which has no parts
- (2) A line is a breadthless length
- (3) The extremities of a line are points
- (4) A straight line is a line that lies evenly with the points on itself
- (5) A surface is that which has length and breadth only

...

(23) Parallel straight lines are lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in any direction.

Euclid's common notions include:

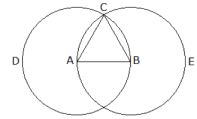
- (1) Things that are equal to the same thing are also equal to one another
- (2) If equals are added to equals, the wholes are equal
- (3) If equals are subtracted from equals, the remainders are equal
- (4) Things that coincide with one another are equal
- (5) The whole is greater than its parts

Euclid's postulates include:

- (1) A straight line can be drawn from any point to any other point

- (2) A finite straight line can be produced continuously in a line
- (3) A circle may be described with any center and distance
- (4) All right angles are equal to one another
- (5) If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, then the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles.

Proposition 1.1: Def. 1.20: An equilateral triangle is that which has its three sides equal, an isosceles triangle has two sides equal, and a scalene triangle has no equal sides. Proposition 1.1: to construct an equilateral triangle on a given straight line.



Euclid organized Book I so as to lead to this result, this specific proof was probably Euclid's own and termed 'Pythagorean theorem', 800 years after Euclid, proposition 1.47. The 'mousetrap proof': often, as in the proposition of Pythagoras, lines are drawn, we don't know why, and it afterwards appears that they were traps which close unexpectedly and take prisoner the assent of the astonished learner (A. Schopenhauer, 1912).

In summary, Euclid's *Elements* is a compilation of Greek mathematics written at the beginning of the Hellenistic period, earliest treatise of theoretical mathematics arrived to us in its entirety, dominant influence on theoretical mathematics for over two millennia, an axiomatic-deductive system (includes definitions, postulates, common notions, propositions, proofs) and model for scientific reasoning.

TEXTBOOK NOTES

CHAPTER 5: HELLENISTIC MATHEMATICS, EUCLID'S *THE ELEMENTS* IN PARTICULAR

5-1 THE GOLDEN AGE OF GREEK MATHEMATICS

The classical Greek period ended after King Philip defeated an alliance and his son created an empire from Egypt to the Indus River. The Greek culture transferred among the Mediterranean world, along with other cultures. But, since the Greek culture was the dominant culture, the period was known as the *Hellenistic Age*. In this time, theoretical mathematics was considered a special discipline among the educated elite. Scholars were interested in systematizing, expanding and preserving the language in writing with its own distinct form of expression and communication.

The most famous cultural institution of the Hellenistic Age was the Museum, a *temple of the Muses*, founded by King Ptolemy I Soter. Philosophers, mathematicians, poets, linguists, historians and others gathered at the Museum to practice their studies in a common dialect. The purpose of their gathering was to expand knowledge for the benefit of humankind. The Museum was used for seven centuries until the decline of the Roman Empire. Since the famous Greek mathematicians: Euclid, Archimedes and Apollonius lived in the Hellenistic Age, it was known as the Golden Age of Greek mathematicians. The works created during this time were the earliest dated mathematical works, such as *The Elements* by Euclid.

5-2 WHO WAS EUCLID, AND WHAT EXACTLY DID HE WRITE?

There are several speculations regarding any information about Euclid, besides his book *The Elements*. Such as, the possibility he studied at Plato's Academy in Athens, he led a school of geometry at the Museum, he may have been acquainted with Archimedes and that he may have authored other books. Other stories about Euclid displays the moral tales conveyed by Euclidean geometry, one, that its not for material gain and for practical rewards and, two, there is no privileged access to it.

There are speculations that Euclid may not have been the only author of *The Elements*, since it was a collection of mathematical knowledge accumulated by Euclid's predecessors. Euclid's book may have sufficiently comprehensive enough to make it the primary source for others, even though it was accumulated from the works of others, who also separately published their literature. During this ancient time, there was no condition for authors to cite their sources

and give credit to others which made it difficult to determine if someone's work was original or taken from someone else. Such was the case for Euclid's book, which however have sufficient evidence to believe that it was not an original creation.

The earliest text of Euclid's *The Elements* dated from 888 CE which cannot be deemed as the original version of the book. The translation of the verses was difficult because any additions (scholia) to the book by translators were considered part of the original work by other translators. Editors and commentators also added their own clarifications and simplifications in translation their work which made it difficult to restore the original version of the book.

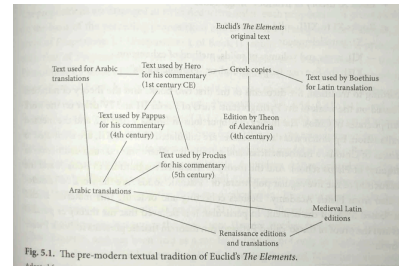


Fig. 5.1. The pre-modern textual tradition of Euclid's *The Elements*.

5-3 EUCLID'S *ELEMENTS*

Euclid's *Elements* was composed of thirteen parts:

- (1) Book I to VI: Plane Geometry
 - (1) First Principles (definitions, postulates, and common notions), triangles, parallel lines, Pythagoras theorem
 - (2) Transformation of Areas
 - (3) Circles
 - (4) Regular Polygons
- (2) Books V and VI: Theory of Proportions
- (3) Books VII to IX: Theory of Numbers
- (4) Book X: Theory of Irrational Magnitudes
- (5) Books XI and XIII: Solid Geometry
 - (1) XI: Parallelepipeds
 - (2) XII: Areas and Volumes of Solids, Method of Exhaustion
 - (3) XII: The Five Regular Polyhedral, or 'Platonic Solids'

The book is organized using a *axiomatic deductive system*, a system of interrelated propositions derived with logical necessity from a set of first principles. There are five postulates and five common notions in Book I, followed by 465 propositions (theorems like QED and QEF). The propositions are arranged in a deductive order, each proposition is proven based on the preceding proposition and of the first principles.

In the nineteenth century, mathematicians were concerned with creating an undisputable logical foundation for their discipline, which required a thorough examination of Euclid's work. They realized that there was room for improvement, especially in regards to the vague definitions. Some argue that the definition and propositions in Euclid's *Elements* was never meant to be a rigorous source of mathematical knowledge, but provide readers with a primary understanding of the mathematical concepts developed at the time.

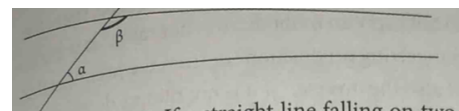
EUCLID'S AXIOMS AND POSTULATES

Euclid's five common notions, also called axioms, are,

- (1) Things that are equal to the same thing are also equal to one another
- (2) If equals are added to equals, the wholes are equal
- (3) If equals are subtracted from equals, the remainders are equal
- (4) Things that coincide with one another are equal
- (5) The whole is greater than its parts

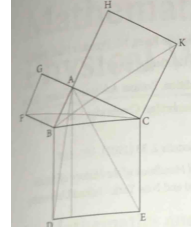
The five geometrical statements chosen by Euclid as postulates for his system are,

- (1) A straight line can be drawn from any point to any other point
- (2) A finite straight line can be produced continuously in a line
- (3) A circle may be described with any center and distance
- (4) All right angles are equal to one another
- (5) If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, then the two



straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles

Axioms must be consistent and not contradict one another, they must be complete and express explicitly everything and be independent. Some criticized Euclid's work (especially proposition 1) for requiring tacit assumptions. Some argued that the fifth postulate was unnecessary because the proposition 1.29 and 1.30 already described a parallel line; however, in Euclid's defense, a proposition and its inverse are independent and cannot prove or disprove each other, so each needs to be proven separately as in the case of these propositions and the fifth postulate. It is believed that from his theory of parallel lines came the so-called Pythagorean theorem.



Other mathematicians questioned Euclid's logical construction and tried to demonstrate that the fifth postulate was a theorem provable on the bases of the first four postulates. From their efforts, alternative geometries were formulated without the fifth postulate known as non-Euclidean geometries.

Lecture 8

NOVEMBER 12, 2018

LECTURE NOTES

EUCLID'S FIFTH POSTULATES TO NON-EUCLIDEAN GEOMETRIES

THE LEGACY OF EUCLID'S *ELEMENTS*

For 2200 years, Euclid's *Elements* was the backbone of mathematical education in the Greco-Roman world, Islam, and Europe. Its axiomatic-deductive structure became the model of a mathematical theory. An axiom-deductive system accepts notions without proof and declares them as axioms/postulates. In the 17th century, the Scientific Revolution established mathematics and experimentalism as the foundations of all western science. To this day, many scientists believe that to be scientific a theory must be formulated as an axiomatic-deductive system.

NEWTON'S RATIONAL MECHANICS

Issac Newton, *The Mathematical Principles of Natural Philosophy*, 1687 was organized in an axiomatic-deductive system. The Books were:

- (1) The Book I: The Motion of Bodies includes 8 definitions, 3 axioms or laws of motion, and 98 propositions and their proofs.
- (2) The Book II: The Motion of Bodies in Resisting Mediums.
- (3) The Book III: The System of the World

In the book, the physical space identified with Euclidean space.

CRITIQUES OF *THE ELEMENTS*

In the industrial age, there were defined academic fields —physics, geometry, philosophy...The goal of mathematics was to determine axioms and conceptual foundations for the axioms. Euclid's geometry determined conceptual foundation that led to non-Euclidean geometries.

From 2nd century CE-1800s, mathematicians uncovered a problem with the fifth postulate, some wondered if Euclid made a mistake. In 1800s, the growth and professionalization of mathematics led to the demand of mathematical rigour and there was a deeper examination of Euclid's geometry. There were uncovered hidden assumptions in Euclid's *Elements*. This meant that the assumptions were not declared, as all axioms and postulates must be in order to be accepted without proof. Examples include continuity, superposition, extensibility of lines to infinity.

Besides hidden assumptions, there was the realization for the need of primitive terms (undefined terms) which made certain assumptions difficult to understand (ex. breadlength definition?). The critique of Euclid's *Elements* led to the discovery of non-Euclidean geometries and separation of mathematics from physics.

THE COMPLETION OF EUCLID'S PROJECT

David Hilbert, *The Foundations of Geometry*, 1899 solved the hidden assumptions and need for primitive terms problem of Euclid's *Elements*. In his book, there were:

- (1) 3 primitive terms: point, line, plane
- (2) 3 primitive notions: between, to lie on, congruence
- (3) 20 axioms

EUCLID'S PARALLEL LINES

Def. I,23: Parallel lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in any direction.

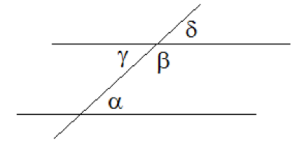
2nd Postulate: A straight line can be produced continuously in a line.

The hidden assumption is that straight lines can be extended to infinity and never return onto themselves. Only with this hidden assumption it can be proven that parallel lines exist.

THE PROBLEM OF EUCLID'S FIFTH POSTULATE

Def. I, 27: If a straight line falling on two straight lines makes a) the alternate angles equal, then two lines are parallel to one another.

Def. I, 28: If a straight line falling on two straight lines makes b) the exterior angle equal to the interior and opposite angle on the same side, or c) the sum of the two interior angles on the same side equal to two right angles, then the two lines are parallel to one another

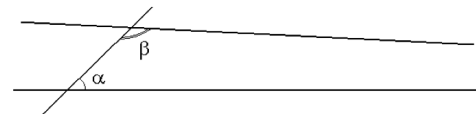


These propositions can be demonstrated from the first 3 postulates only, plus the Common Notions and the assumption that lines can be extended to infinity.

Def. I, 29: A straight lines falling on parallel lines makes a) the alternate angles equal to one another, b) the exterior angle equal to the interior and opposite angle, c) and the sum of the interior angles on the same side equal to two right angles.

This proposition requires the 5th postulate to be demonstrated.

5th Postulate: If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, then the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles.



This postulate was not as simple and compact as the other postulates and the first 28 theorems did not depend on it. Shouldn't it be a theorem instead of a postulate?

- Proposition 27 and 28 are direct: If P then Q
- Proposition 29 is the converse of them: If Q then P
- Postulate 5 is the inverse of the direct: If not P then not Q

Euclid was right all along because there needed to be a postulate in order to create Proposition 29, which could not be proven from Proposition 28 (like all other propositions). A converse (29) and proposition (28) are not logically equivalent/related to one another, but the inverse (5th postulate) and converse (29) are related to one another.

In summary:

<p>I, 27 and I, 28:</p> <ul style="list-style-type: none"> • If <ol style="list-style-type: none"> $\alpha = \gamma$ $\alpha = \delta$ $\alpha + \beta = 2R$ • then l and l' are parallel 	<p>I, 29</p> <ul style="list-style-type: none"> • If l and l' are parallel • then <ol style="list-style-type: none"> $\alpha = \gamma$ $\alpha = \delta$ $\alpha + \beta = 2R$
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Postulates 1-4
Tacit assumption of infinite extensibility

if $\alpha + \beta = 2$ right angles, then l and l' are parallel (Proposition I.28)

if l and l' are parallel, then $\alpha + \beta = 2$ right angles (Proposition I.29)

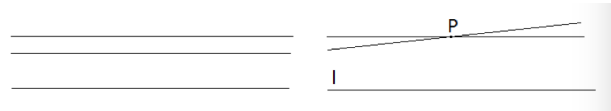
I,29 is not logically equivalent to I,27-28 but to its inverse!



if $\alpha + \beta \neq 2$ right angles, then l and l' are not parallel (inverse of I.28)

The first 4 postulates were tacit assumptions of infinite extensibility with led to proposition 28, but the 5th postulate said that if alpha and beta do not create 2 right angles, then the two lines are not parallel, which led to proposition 29.

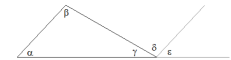
Def. I, 30: *Two lines parallel to the same line are parallel to one another.*



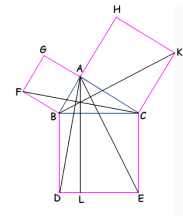
The corollary: hence, given a line I and a point P not on I, on the plane defined by I and P there is one and only one parallel to I through P.

Proof: if there were two lines parallel to I through P, they would have to parallel to one another; but this would be absurd because the two lines meet at P.

Def. I, 32: *In any triangle, if one of the sides is produced, then the exterior angle equals the sum of the interior and opposite angles, and the sum of the interior angles is equal to two right angles.*



Def. I, 47: *In right-angled triangles the square on the hypotenuse is equal to the sum of the squares on the two short sides.*

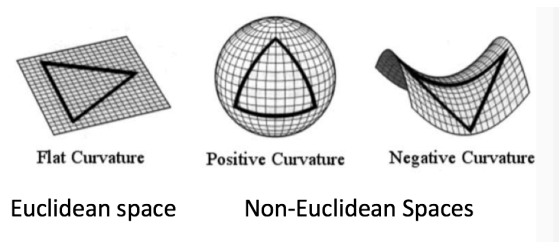


NON-EUCLIDEAN GEOMETRY (1810-1830)

Mathematicians included: Carl Friedrich Gauss, Nikolai Lobachevsky and Janos Bolyai.

- (1) Euclid was correct: the 5th Postulate is a postulate
- (2) Replacing the 5th Postulate with the infinite parallels hypothesis does not lead to any contradiction but to a new geometry different from Euclid's but just as rigorous: —infinite # of parallel lines through a given point and sum of inner angles of a triangle is less than 2 right angles.

Georg Bernhard Riemann (ca. 1850) introduced a new definition of straight line: shortest distance between two points and the idea of the curvature of space, as opposed to Euclid's geometry which had zero curvature and 'flat' space. There was the creation the infinite parallels hypothesis, introducing negative curvature and no parallel hypothesis, with positive curvature.



Which geometry is the one that correctly describes physical space?

It depends on what the curvature of physical space is, it is to be determined empirically and is no longer a question that concerns mathematicians.