Question 1 **[10 marks]**

A miner is at the center of a maze. Two doors lead out of the maze.

- If the miner chooses Door 1, he/she returns back to the maze after 1 minute of scampering;
- If the miner chooses Door 2, he/she always has 0.5 probability getting out of the maze in 1 minute and 0.5 probability getting out of the maze in 2 minutes.

Suppose the miner always chooses Door 1 with probability 1/2 and Door 2 with probability 1/2. Let X = number of minutes for the miner to get out of the maze.

(a) (3pts) Find $E(X)$.

(b) (4pts) Find $E(X^2)$ and $Var(X)$.

(c) (3pts) Find $P(X = 2)$.

Hint: conditional on Y, where

$$Y = \begin{cases} 1, & \text{if miner chooses Door 1 at the first time,} \\ 2, & \text{if miner chooses Door 2 at the first time and gets out of the maze in 1 minute,} \\ 3, & \text{if miner chooses Door 2 at the first time and gets out of the maze in 2 minutes.} \end{cases}$$

Sol:

$$X = \begin{cases} 1+R & Y=1 \\ 1 & Y=2 \\ 2 & Y=3 \end{cases}$$

R = Remaining time that the miner needs to spend for getting out of the maze when he/she returns back to the maze

X & R have the same distribution

$P(Y=1)=\frac{1}{2}$; $P(Y=2)=\frac{1}{4}$; $P(Y=3)=\frac{1}{4}$

(a) $E(X) = E[E(X|Y)]$

$= E(X|Y=1)P(Y=1) + E(X|Y=2)P(Y=2) + E(X|Y=3)P(Y=3)$ — 0.5

$= \underbrace{\frac{1}{2}E(1+R)}_{1pt} + \underbrace{\frac{1}{4}*1 + \frac{1}{4}*2}_{0.5pt}$ — 1.5

$= \frac{1}{2} + \frac{1}{2}\underbrace{E(R)}_{=E(X)} + \frac{3}{4}$

$\Rightarrow E(X) = \frac{1}{2} + \frac{1}{2}E(X) + \frac{3}{4} \Rightarrow E(X) = \frac{5}{2}$ — 1

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(b) $E(X^2) = E[E(X^2|Y)]$

$= E(X^2|Y=1)P(Y=1) + E(X^2|Y=2)P(Y=2) + E(X^2|Y=3)P(Y=3)$ — 0.5

$= \underline{\frac{1}{2}E[(1+R)^2]} + \frac{1}{4}*1 + \frac{1}{4}*2^2$ — 1.5

$= \frac{1}{2}E(1+2R+R^2) + \frac{5}{4}$

Note $E(R) = E(X) = \frac{5}{2}$

$E(R^2) = E(X^2)$

$\Rightarrow$

$E(X^2) = \frac{6}{2} + \frac{1}{2}E(X^2) + \frac{5}{4}$

$\Rightarrow E(X^2) = \frac{17}{2}$ — 1

Hence $Var(X) = \frac{17}{2} - \left(\frac{5}{2}\right)^2 = \frac{17}{2} - \frac{25}{4} = \frac{9}{4}$ — 1

(c) $P(X=2)$

$= \underline{P(Y=3)}$ 1 $+ \underline{P(Y=1 \text{ \& miner chooses Door 2 at 2nd time and gets out of maze in 1 minute})}$ 1

$= \frac{1}{4} + \frac{1}{2} * \frac{1}{4} = \frac{3}{8}$ — 1

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Question 2 **[10 marks]**

Suppose $X_1, X_2, X_3, \ldots$ are a sequence of independent random variables such that for $n \geq 1$,

$$X_n \sim Bin(2, 0.5^n).$$

That is, $X_1 \sim Bin(2, 0.5)$, $X_2 \sim Bin(2, 0.5^2)$, $X_3 \sim Bin(2, 0.5^3)$, and so on. Assume $N \sim Geo(0.5)$. Further N and $X_1, X_2, X_3, \ldots$ are independent.

(a) (3pts) Find $E(X_N)$.

(b) (4pts) Find $E(X_N^2)$ and $Var(X_N)$.

(c) (3pts) Find $P(X_N = 2)$.

Hint: for (a)–(c), conditional on N.
Note: $\sum_{k=1}^{\infty} s^k = \frac{s}{1-s}$.

Sol:

(a) $E(X_N) = E[E(X_N|N))$

X_n & N are independent

$E(X_N|N=n) = \underbrace{E(X_n|N=n)}_{\text{substitution rule}} = E(X_n) = 2*0.5^n$ 1

Hence $E(X_N|N) = 2*0.5^N$ -0.5 if E(·) wrong

and $E(X_N) = E[2*0.5^N]$ 0.5 $= \sum_{n=1}^{\infty} 2*0.5^n * \overset{0.5^n}{P(N=n)}$ 1 $= \sum_{n=1}^{\infty} 2*(0.25)^n$

$= 2 * \frac{0.25}{1-0.25} = \frac{2}{3}$ 0.5

(b) $E(X_N^2) = E[E(X_N^2|N)]$

X_n & N are independent

$E(X_N^2|N=n) = \underbrace{E(X_n^2|N=n)}_{\text{substitution rule}} = E(X_n^2) = Var(X_n) + E^2(X_n)$ 1

$= 2*0.5^n(1-0.5^n) + 4*(0.25)^n$

$= 2*0.5^n + 2*0.25^n$ 1

$\Rightarrow E(X_N^2|N) = 2*0.5^N + 2*0.25^N$

$E(X_N^2) = E[2*0.5^N + 2*0.25^N] = \sum_{n=1}^{\infty}(2*0.5^n + 2*0.25^n)*0.5^n$

$= \frac{2}{3} + \frac{2}{7}$ 1

$(= \frac{20}{21})$

$\Rightarrow E(X_N^2) = \frac{2}{3} + \frac{2}{7}$

$Var(X_N) = \frac{2}{3} + \frac{2}{7} - \frac{4}{9} = \frac{2}{9} + \frac{2}{7}$ 1 $(\frac{32}{63})$

Please initial:

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1c)

$$P(X_N=2) = \sum_{n=1}^{\infty} P(X_N=2 \mid N=n) * P(N=n) \quad 0.5$$

$$= \sum_{n=1}^{\infty} P(X_n=2 \mid N=n) * 0.5^n$$

$$= \sum_{n=1}^{\infty} P(X_n=2) * 0.5^n \quad 1$$

$$= \sum_{n=1}^{\infty} [0.5^n]^2 * 0.5^n \quad 1$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{8}\right)^n$$

$$= \frac{1}{7} \quad 2.5$$

Question 3 **[10 marks]**

Suppose $X_1, X_2, \ldots$ are a sequence of independent and identically distributed uniform random variables with probability density function

$$f(x) = \begin{cases} 1, & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let Y be a continuous random variable with probability density function

$$g(y) = \begin{cases} 3y^2, & 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

We further assume that Y and $X_1, X_2, \ldots$ are independent. Let $T = \min\{n \geq 1 : X_n < Y\}$. That is, T is the first time to get a smaller value than Y in the sequence $\{X_n\}_{n=1}^{\infty}$.

(a) (3pts) Find $P(T = 2) = P(X_1 \geq Y, X_2 < Y)$.

(b) (3pts) Find $E(T)$.

(c) (4pts) Find $E(T^2)$ and $Var(T)$.

Hint: conditional on Y for (a)–(c). For (b) and (c), further note that $T|Y = y \sim Geo\big(p(y)\big)$. Figure out the form of $p(y)$ in terms of y.

Sol: (a) $P(T=2) = P(X_1 \geq Y, X_2 < Y)$

$= \int_0^1 P(X_1 \geq Y, X_2 < Y \mid Y=y) f_Y(y)\, dy$ 0.5

substitution rule

$= \int_0^1 P(X_1 \geq y, X_2 < y \mid Y=y)\, 3y^2\, dy$ 0.5

X_1, X_2 & Y are independent

$= \int_0^1 P(X_1 \geq y, X_2 < y)\, 3y^2\, dy$

1pt

$= \int_0^1 P(X_1 \geq y) P(X_2 < y)\, 3y^2\, dy$

X_1 & X_2 are independent

$P(X_1 \geq y) = \int_y^1 1\, dx = (1-y)$

$P(X_2 < y) = y$

1pt

$= \int_0^1 (1-y)\, y * 3y^2\, dy$

$= \int_0^1 3y^3 dy - \int_0^1 3y^4 dy = \frac{3}{4} - \frac{3}{5} = \frac{3}{20}$

Please initial:

1b)

$T|Y=y \sim \text{Geo}(y)$

Given $Y=y$, T = 1st time to get a smaller value than y in $\{X_n\}_{n=1}^{\infty}$

$p(X_i<y)=y$

$E(T|Y=y)=\frac{1}{y}$

0.5 2

$\Rightarrow E(T)=E(E(T|Y))=\int_0^1 E(T|Y=y)f_Y(y)dy$

$=\int_0^1 \frac{1}{y}3y^2 dy = \int_0^1 3y\,dy = \frac{3}{2}$

0.5

1c)

0.5

$E(T^2)=E(E(T^2|Y))=\int_0^1 E(T^2|Y=y)f_Y(y)dy$

$E(T^2|Y=y)=\left(\frac{1}{y}\right)^2+\frac{1-y}{y^2}$

$T|Y=y\sim \text{Geo}(y)$

2pt

$=\int_0^1\left(\frac{2-y}{y^2}\right)*3y^2dy$

$=\int_0^1 3(2-y)\,dy = 6-\frac{3}{2}=\frac{9}{2}$

0.5

$\text{Var}(T)=\frac{9}{2}-\frac{9}{4}=\frac{9}{4}$ 1