1.(a)(6 marks) For the differential equation $\frac{dy}{dx} = -y^2$, verify that all members of the family $y = 1/(x+C)$ with arbitrary constant C are solutions of the given equation.

(b)(6 marks) Find a solution of the initial value problem

$$\frac{dy}{dx} = -y^2, \quad y(0) = 0.5$$

(a) If $y = \frac{1}{x+C}$

then $\frac{dy}{dx} = -\frac{1}{(x+C)^2}$ / 3 marks for y'

substituting them into the equation.

$LHS = \frac{dy}{dx} = -\frac{1}{(x+C)^2} = -\left(\frac{1}{x+C}\right)^2 = -y^2 = RHS$ / 3 marks

(b) If $y = \frac{1}{x+C}$

then $y(0) = \frac{1}{0+C} = \frac{1}{C} = 0.5$

so $C = 2$ / 4 marks for C

therefore the solution is $y = \frac{1}{x+2}$ / 2 marks for the final answer.

2.(13 marks) Find the orthogonal trajectories of the family of curves $y = ke^x$ where k is an arbitrary constant. Describe these orthogonal trajectories.

A differential equation that is satisfied by $y = ke^x$ is

$$\frac{dy}{dx} = ke^x = y$$

so the slope of the tangent line at any point (x, y) on the curve is

$y' = y$ / 3 marks.

For the orthogonal trajectories, the slope must be

$y' = -\frac{1}{y}$ (the negative reciprocal of the slope) / 3 marks

then we have the separable differential equation

$$\frac{dy}{dx} = -\frac{1}{y}$$

Solving the differential equation

$$\int y \, dy = -\int dx$$

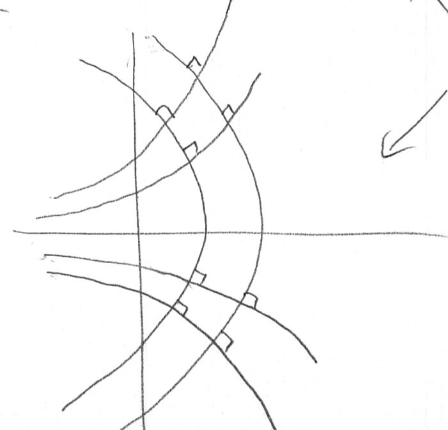
$\frac{y^2}{2} = -x + C$ or $x = -\frac{y^2}{2} + C$ where C is a constant

/ 4 marks

The orthogonal trajectories are parabolas symmetric about the x-axis and open to the left

either explanation (in words) or graph

Both are o.k.



/ 3 marks

3.(12 marks) Determine whether the sequence is convergent or divergent. If it is convergent, find the limit. Give a reason for your answer.

$$a_n = \frac{\sin(n^2+1)}{n+1}$$

since $0 \le \left| \frac{\sin(n^2+1)}{n+1} \right| \le \frac{1}{n+1}$ / 4 marks

and $\lim_{n\to\infty} 0 = \lim_{n\to\infty} \frac{1}{n+1} = 0$, / 3 marks

then $\lim_{n\to\infty} |a_n| = 0$

$\therefore \lim_{n\to\infty} a_n = 0$ / 3 marks

and the sequence is convergent / 2 marks

(Alternatively,

since $-\left(\frac{1}{n+1}\right) \le \frac{\sin(n^2+1)}{n+1} \le \frac{1}{n+1}$

and $\lim_{n\to\infty} -\left(\frac{1}{n+1}\right) = \lim_{n\to\infty} \frac{1}{n+1} = 0$,

then $\lim_{n\to\infty} a_n = 0$)

4.(13 marks) The given curve is rotated about the y-axis. Find the area of the resulting surface.

$$x^{2/3} + y^{2/3} = 1, \quad 0 \le y \le 1$$

The curve is symmetric about the y-axis from $x=-1$ to $x=1$, so we use the portion of the curve from $x=0$ to $x=1$

Consider the formula

$$S = \int_0^1 2\pi x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$ / 3 marks

to compute $\frac{dy}{dx}$,

$$y^{2/3} = 1 - x^{2/3} \Rightarrow y = \left(1 - x^{2/3}\right)^{3/2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{3}{2}\left(1 - x^{2/3}\right)^{1/2}\left(-\frac{2}{3}x^{-1/3}\right) = -\frac{\sqrt{1-x^{2/3}}}{x^{1/3}}$$ / 3 marks

then $1+\left(\frac{dy}{dx}\right)^2 = 1 + \frac{1-x^{2/3}}{x^{2/3}} = x^{-2/3}$ / 2 marks

therefore,

$$S = \int_0^1 2\pi x \left(x^{-1/3}\right) dx$$ / 2 marks

$$= 2\pi \int_0^1 x^{2/3}\, dx = 2\pi \left(\frac{3}{5}x^{5/3}\right)\Big|_0^1$$

$$= 2\pi\left(\frac{3}{5}\right) = \frac{6\pi}{5}$$ / 3 marks

(Alternatively, the formula $S = \int 2\pi x \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$ can be used)