4.4 The ABCD Law

→ transformation of the Gaussian beam…

…after propagating through any number of lenslike media and elements

→ need to return to our original formulation for $g \neq 0$

→ constraint equations (4.12) and (4.13)

Let us consider the former…

$$\frac{\partial}{\partial z}\left(\frac{1}{q(z)}\right)+\frac{1}{q^2(z)}+g^2=0 \qquad (4.55)$$

…and again introduce, as in (4.15), the change of variables

$$\frac{1}{q(z)}=\frac{1}{u}\frac{\partial u}{\partial z} \qquad (4.56)$$

(4.56) into (4.55) yields…

$$\frac{\partial^2 u}{\partial z^2}+g^2 u=0 \qquad (4.57)$$

↑ a simple harmonic oscillator equation

…which has the solution:

$$u(z)=a\sin(gz)+b\cos(gz) \qquad (4.58)$$

(4.58) into (4.56), using $q(0) = q_0$ to set a and b, yields

$$q(z)=\frac{q_0\cos(gz)+g^{-1}\sin(gz)}{-gq_0\sin(gz)+\cos(gz)} \qquad (4.59)$$

↑ describes the evolution of the Gaussian beam parameter

NB. For a homogeneous medium, $g = 0$ and (4.59) reduces to (4.18).

Comparing (4.59) with (3.47), the matrix equation for the ray trajectory in a lenslike medium, reveals that the media transform q_1 into q_2 as…

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D} \qquad (4.60)$$

the ABCD law: a bilinear (Möbius) transformation

…where A, B, C, D are the elements of the ray matrix that relate the ray (ρ, ρ') at plane 2 to the ray at plane 1.

NB. This is an elementary conformal mapping

→ for real matrix elements, if q_1 is in the upper complex plan, then so is q_2

i.e., the beam will remain Gaussian and confined

If we consider a thin lens of focal length f, whose ray matrix is shown in (3.3), then we obtain

$$\frac{1}{q_2} = \frac{1}{q_1} - \frac{1}{f} \qquad (4.61)$$

transformation of a Gaussian beam by a thin lens

(4.51) into (4.61), considering the real and imaginary parts independently, yields

$$\frac{1}{R_2} = \frac{1}{R_1} - \frac{1}{f} \qquad (4.62)$$

$$\omega_2 = \omega_1 \qquad (4.63)$$

NB. The above equations (4.61) – (4.63), also hold for the transformation of a Gaussian beam by the reflection from a mirror whose radius of curvature is R if $f \rightarrow R/2$.

- **Adjacent Lenslike Media**

  - Described by $\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}$ and $\begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}$

If q_1 is the input to medium 1, then…

$$q_2 = \frac{A_1 q_1 + B_1}{C_1 q_1 + D_1}, \quad q_3 = \frac{A_2 q_2 + B_2}{C_2 q_2 + D_2}$$

q_2 → output from medium 1 and input to medium 2

…can be combined to yield

$$q_3 = \frac{A_T q_1 + B_T}{C_T q_1 + D_T} \qquad (4.64)$$

q_3 → output from medium 2

…where the matrix relating the output plane to the input plane is

$$\begin{bmatrix} A_T & B_T \\ C_T & D_T \end{bmatrix} = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \qquad (4.65)$$

- **Generalization**

  → any arbitrary number of lenslike elements and media

  → The transformation of the Gaussian beam from input plane to output plane is the ordered product of the matrices for each element in the chain

$R(z)$ and $\omega(z)$ can be recovered at any z from (4.51)

e.g., Gaussian Beam Focusing

Consider a Gaussian beam, with a minimum spot size of ω_{01}, incident on a thin lens of focal length f. The waist of the incident beam is a distance d_1 in front of the lens. Where is the waist of the output beam and what is its spot size ω_{02} at that point?

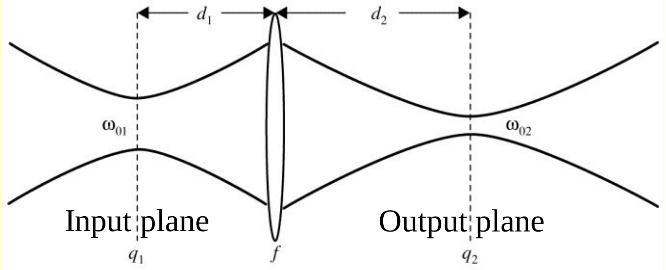

Fig. 4.7
Focusing of a Gaussian beam.
Yariv & Yeh, 6th ed., p. 85

For the input plane: $\omega_1 = \omega_{01}\,,\quad R_1 \to \infty$

For the output plane: $\omega_2 = \omega_{02}\,,\quad R_2 \to \infty$

From (4.51), the Gaussian beam parameters are…

$$\frac{1}{q_1} = \frac{-i\lambda}{\pi n \omega_{01}^2} = \frac{1}{iz_1} \qquad (4.66)$$

$$\frac{1}{q_2} = \frac{-i\lambda}{\pi n \omega_{02}^2} = \frac{1}{iz_2} \qquad (4.67)$$

(4.66) is related to (4.67) via the ABCD law of (4.60)…

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D} \qquad (4.68)$$

…where the matrix elements are found from

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & d_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} 1 & d_1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - d_2/f & d_1 + d_2 - d_1 d_2/f \\ -1/f & 1 - d_1/f \end{bmatrix} \qquad (4.69)$$

(4.66) and (4.67) in (4.68) yield

$$iz_2 = \frac{Aiz_1 + B}{Ciz_1 + D} = \frac{ACz_1^2 + BD + iz_1\left(AD - BC\right)}{C^2 z_1^2 + D^2}$$

Since the ray matrix is unimodular, $AD - BC = 1$, so

$$iz_2 = \frac{\left(ACz_1^2 + BD\right) + iz_1}{C^2 z_1^2 + D^2} \qquad (4.70)$$

(4.70) imposes constraints on the real and imaginary parts as

$$ACz_1^2 + BD = 0 \qquad (4.71)$$

$$z_2 = \frac{z_1}{C^2 z_1^2 + D^2} \qquad (4.72)$$

(4.71) into (4.72) yields (again using unimodularity)…

$$z_2 = \frac{z_1}{D^2 - BDC/A} = \frac{A}{D} z_1 \qquad (4.73)$$

(4.69) into (4.73) yields…

$$z_2 = \left(\frac{f - d_2}{f - d_1}\right) z_1 \qquad (4.74)$$

…and (4.66) and (4.67) into (4.74) then show:

$$\omega_{02}^2 = \left(\frac{f - d_2}{f - d_1}\right) \omega_{01}^2 \qquad (4.75)$$

this relates the two spot sizes

The location of the beam output waist can be found from (4.69) into (4.71) as…

$$z_1^2 = \frac{\left(d_1 f + d_2 f - d_1 d_2\right)\left(d_1 - f\right)}{d_2 - f} \qquad (4.76)$$

Writing the output location in terms of the input yields, after some algebra…

$$d_2 - f = \frac{f^2 \left(d_1 - f\right)}{z_1^2 + \left(d_1 - f\right)^2} \qquad (4.77)$$

(4.77) into (4.75) yields the minimum output beam spot size as…

$$\omega_{02}^2 = \frac{f^2 \omega_{01}^2}{z_1^2 + (d_1 - f)^2} \qquad (4.78)$$

…where z_1 is found from (4.66).

NB. For a point source, $z_1 \rightarrow 0$ and (4.77) simplifies to…

$$d_2 - f = \frac{f^2}{d_1 - f}$$

…which can be written more familiarly as

$$\frac{1}{d_1} + \frac{1}{d_2} = \frac{1}{f} \qquad (4.79)$$

thin lens equation

NB. A plane wave can be considered as a Gaussian beam of infinite spot size. Thus, for such an input, $z_1 \rightarrow \infty$ and (4.77) yields

$d_2 = f$ ← the beam waist at the focal point

NB. If the input Gaussian beam has its waist at the front focal point ($d_1 = f$), then the output Gaussian beam has its waist at the rear focal point ($d_2 = f$). Then (4.69) reduces to…

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 0 & 2f - 1 \\ -1/f & 0 \end{bmatrix}$$

…and (4.72) becomes

$$z_1 z_2 = f^2 \qquad (4.81)$$

…which, using (4.66) and (4.67), give the waist radii as

$$\omega_{01}\omega_{02} = \frac{\lambda f}{\pi n} \qquad (4.82)$$

NB. If the input Gaussian beam has its waist located at the lens, then $d_1 = 0$ and (4.77) gives…

$$d_2 = \frac{f}{1+\left(f/z_1 \right)^2} \qquad (4.83)$$

while (4.78) gives…

$$\frac{\omega_{02}}{\omega_{01}} = \frac{f/z_1}{\sqrt{1+\left(f/z_1 \right)^2}} \qquad (4.84)$$

4.5 Higher-Order Gaussian Modes

If we now remove the radial symmetry condition, the Helmholtz equation (4.1) will have solutions that possess azimuthal dependence.

Let us restrict ourselves to a homogeneous medium.

In Cartesian coordinates, the solutions are…

$$E_{l,m}(x,y,z) = E_0 \frac{\omega_0}{\omega(z)} H_l\left(\frac{\sqrt{2}x}{\omega(z)}\right) H_m\left(\frac{\sqrt{2}y}{\omega(z)}\right) \times \exp\left(\frac{-ik\left(x^2+y^2\right)}{2q(z)} - i\theta\right) \qquad (4.85)$$

Hermite-Gaussian beam of mode order (*l*,*m*)

Hermite polynomial of order *m*

where the on-axis phase shift is…

$$\theta = kz - (l+m+1)\arctan\left(z/z_0\right) \qquad (4.86)$$

with (4.24) providing $z_0 = \pi n \omega_0^2 / \lambda$, and

$\omega(z) -$ defined in (4.29)

$R(z) -$ defined in (4.30)

$\eta(z) -$ defined in (4.31)

$q(z) -$ defined in (4.51)

Defining…

$$\xi_x = \sqrt{2}x \big/ \omega, \quad \xi_y = \sqrt{2}y \big/ \omega \qquad (4.87a)$$

(4.85) may be written in terms of the Hermite-Gaussian functions

$$u_l(\xi) = \left(l!2^l\sqrt{\pi}\right)^{-1/2} H_l(\xi) e^{-\xi^2/2} \qquad (4.87b)$$

as:

$$E_{l,m}(x,y,z) = E_0\left[u_l(\xi_x)u_m(\xi_m)\right]\left[\left(\frac{\omega_0}{\omega(z)}\right)e^{-i\theta(z)}\right] \qquad (4.88)$$

transverse variation ⟶ $\left[u_l(\xi_x)u_m(\xi_m)\right]$

longitudinal variation ⟶ $\left[\left(\frac{\omega_0}{\omega(z)}\right)e^{-i\theta(z)}\right]$

NB. The Hermite-Gaussian beams form a complete set of orthogonal functions satisfying…

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_{l_1,m_1}(x,y,z)E_{l_2,m_2}(x,y,z)\,dxdy = \delta_{l_1,l_2}\delta_{m_1,m_2} \qquad (4.89)$$

…and hence they can be used as a basis to described any paraxial optical beam as

weighting coefficients ⟶ $c_{l,m}$

$$E(x,y,z) = \sum_l \sum_m c_{l,m}\, E_{l,m}(x,y,z) \qquad (4.90)$$

An equivalent solution to (4.1) may be expressed in cylindrical coordinates as

$$E_{p,m}(\rho,\phi,z) = E_0 \exp\left(\frac{-\rho^2}{\omega^2(z)}\right)\left(\frac{\omega_0}{\omega(z)}\right)\left(\frac{\sqrt{2}\rho}{\omega(z)}\right)^{|m|} L_p^{|m|}\left(\frac{2\rho^2}{\omega^2(z)}\right)$$

$$\times \exp\left(\frac{-ik\rho^2}{2q(z)} - ikz + im\phi + i\left(2p+|m|+1\right)\eta(z)\right) \qquad (4.91)$$

$E_{p,m}$ ⟶ Laguerre-Gaussian beam of order (*p*,*m*) ⟵ mutually orthogonal

L_p^m – associated Laguerre polynomial

$p \geq 0$ – radial index; *m* – azimuthal index

  - radial dependencies of beams with $e^{\pm im}$ are identical
  - intensity patterns are cylindrically symmetric
- useful when the problem has cylindrical symmetry

NB. The fundamental modes in both representations (l = 0, m = 0) and (p = 0, m = 0), are identical…and are what have previously been presented in (4.34) and (4.52).

NB. The following is a list of the first few Hermite and associated Laguerre polynomials…

$$H_0(x) = 1 \quad H_1(x) = 2x \quad H_2(x) = 4x^2 - 2 \quad H_3(x) = 8x^3 - 12x$$

$$H_4(x) = 16x^4 - 48x^2 + 12 \quad L_0^m(x) = 1 \quad L_1^m(x) = -x + (m+1)$$

$$L_2^m(x) = \tfrac{1}{2}\left[x^2 - 2(m+2)x + (m+1)(m+2)\right] \quad L_3^m(x) = \tfrac{1}{6}\left[-x^3 + \cdots\right.$$

$$\left.\cdots 3(m+3)x^2 - 3(m+2)(m+3)x + (m+1)(m+2)(m+3)\right]$$

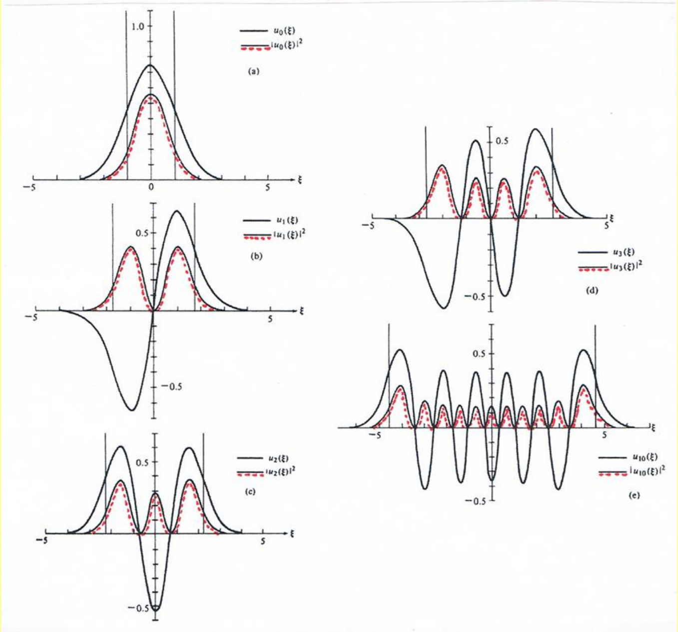

Fig. 4.8 Hermite-Gaussian functions $u_l(\xi)$, for l = 0, 1, 2, 3, and 10, normalized using

$$\int_{-\infty}^{\infty} u_l^2(\xi)\, d\xi = 1$$

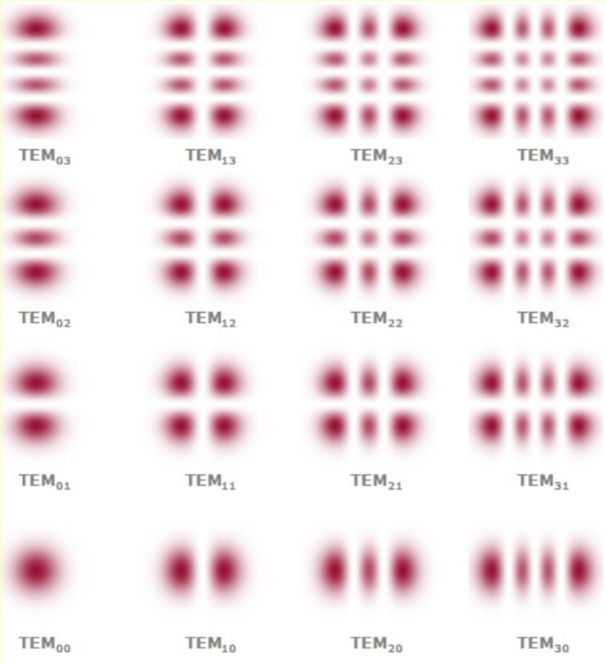

Fig. 4.9 Intensity profiles of the lowest-order Hermite-Gaussian modes. The two mode indices indicate the number of zero crossings of the intensity distributions in horizontal and vertical direction, respectively. From RP Photonics Encyclopedia

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4.6 Gaussian Beam Modes in Quadratic Index Media

Let us return to (4.8), but remove the presumption of an implicit symmetry about the propagating direction

→ in cylindrical coordinates (ρ,ϕ,z), we now explicitly include the azimuthal component ϕ.

First use Cartesian coordinates. Then (4.8) becomes…

$$\nabla^2\mathbf{E}+k^2\left[1-g^2\left(x^2+y^2\right)\right]\mathbf{E}=0 \qquad (4.92)$$

and we now take this to have scalar solutions of the form…

$$E\left(x,y,z\right)=\psi\left(x,y\right)e^{i\beta z} \qquad (4.93)$$

- where β is the propagation constant.

(4.93) into (4.92) yields, using (4.4)…

$$\nabla_t^2\psi+\left[k^2-\beta^2-k^2g^2\left(x^2+y^2\right)\right]\psi=0 \qquad (4.94)$$

$$\partial^2/\partial x^2+\partial^2/\partial y^2$$

Inspection of (4.94) reveals it to be separable. Let…

$$\psi\left(x,y\right)=U\left(x\right)V\left(y\right) \qquad (4.95)$$

(4.95) into (4.94) yields, after division by UV…

$$\frac{1}{U}\frac{d^2U}{dx^2}+\frac{1}{V}\frac{d^2V}{dx^2}+\left[k^2-\beta^2-k^2g^2\left(x^2+y^2\right)\right]=0 \qquad (4.96)$$

Separation of variables implies…

$$\frac{1}{U}\frac{d^2U}{dx^2}+\left(k^2-\beta^2-k^2g^2x^2\right)=C \qquad (4.97)$$

$$\frac{1}{V}\frac{d^2V}{dx^2}+k^2g^2y^2=-C \qquad (4.98)$$

…where C is a *particular* constant.

Let us define: $\xi=\alpha y$ (4.99)

where: $\alpha=\sqrt{gk}$ (4.100)

We may therefore rewrite (4.98) as

$$\frac{d^2V}{d\xi^2}+\left(C/\alpha^2-\xi^2\right)V=0 \qquad (4.101)$$

(4.101) is in the form of the Schrödinger's equation for a simple harmonic oscillator. Its solution is…

$$C/\alpha^2=2m+1,\quad m=1,2,3,\ldots \qquad (4.102)$$

$$V_m\left(\xi\right)=H_m\left(\xi\right)e^{-\xi^2/2} \qquad (4.103)$$

Hermite polynomial of order m

The same procedure may be repeated for (4.97) using…

$$\zeta = \alpha x \tag{4.104}$$

Hence: $$\frac{d^2U}{d\zeta^2} + \left[\left(\frac{k^2-\beta^2-C}{\alpha^2}\right) - \zeta^2\right]U = 0 \tag{4.105}$$

for which: $$\frac{k^2-\beta^2-C}{\alpha^2} = 2l+1, \quad l = 1, 2, 3, \ldots \tag{4.106}$$

$$U_l(\zeta) = H_l(\zeta)\, e^{-\zeta^2/2} \tag{4.107}$$

(4.103) and (4.107) into (4.95) yield…

$$\psi(x,y) = H_l\left(\frac{\sqrt{2}x}{\omega_0}\right) H_m\left(\frac{\sqrt{2}y}{\omega_0}\right) \exp\left[-\left(x^2+y^2\right)\big/\omega_0^2\right] \tag{4.108}$$

where (4.99) and (4.100) permit the definition of the *spot radius*…

$$\omega_0 = \frac{\sqrt{2}}{\alpha} = \sqrt{\frac{2}{gk}} = \sqrt{\frac{\lambda}{g n_0 \pi}} \tag{4.109}$$

NB. (4.109) = (4.32) for $z_0 = g^{-1}$

The propagation constant β is found by solving (4.106), using (4.100) and (4.102), as

$$k^2 - \beta^2 - \alpha^2(2m+1) = \alpha^2(2l+1)$$

$$\beta_{l,m} = k\sqrt{1 - \left({}^{2g}\!/\!{}_{k}\right)(l+m+1)} \tag{4.110}$$

…where *l,m* are mode indices

(4.108) into (4.93) yields the propagating modes as

$$E_{l,m}(x,y,z) = E_0 H_l\left(\frac{\sqrt{2}x}{\omega_0}\right) H_m\left(\frac{\sqrt{2}y}{\omega_0}\right) \exp\left[\frac{-\left(x^2+y^2\right)}{\omega_0^2}\right] \exp\left(-i\beta_{l,m} z\right) \tag{4.111}$$

NB. Unlike (4.34), the spot size has no *z* dependence, due to the focusing effect of the index variation.

The corresponding solutions for higher order modes in a homogeneous medium are rather messy, with all the z dependencies earlier noted, and now Hermite polynomial factors and additional phase terms in the mode indices.

The Hermite polynomials may be defined by a recurrence relation…

$$H_{l+1}(x) = 2xH_l(x) - 2lH_{l-1}(x) \qquad (4.112)$$

where:

$$H_0(x) = 1 \qquad (4.113)$$

$$H_1(x) = 2x \qquad (4.114)$$

NB. The dependence of β on (l,m) causes the modes to have differing phase velocities…

$$v_{l,m} = \omega / \beta_{l,m} \qquad (4.115)$$

and differing group velocities…

$$\left(v_g\right)_{l,m} = d\omega / d\beta_{l,m} \qquad (4.116)$$

NB. (4.92) may also be solved in cylindrical coordinates. The resulting solution will be given in terms of associated Laguerre polynomials.