STUDY! STUDY!

— SOL — TEST – 2 —

STUDY!! Ⓐ

STUDY

1. A sandwich is taken out of the refrigerator, which has a constant temperature of 5 °C, at 10:00 and left on the counter. The temperature in the kitchen is constant at 21 °C. One hour later, the temperature of the sandwich is 12 °C. Let $T(t)$ be the temperature of the sandwich t hours after 10:00.
(i) If the temperature of the sandwich follows Newton's Law of Heating, what is the differential equation that $T(t)$ satisfies?
(ii) Solve the differential equation and give an explicit formula for $T(t)$.
(iii) What is the temperature of the sandwich at noon?

i) $$\boxed{\frac{dT}{dt} = -k\,(T-21)}$$

ii) separate variables $\quad \frac{dT}{T-21} = -k\,dt$

integrate $\quad \int \frac{dT}{T-21} = \int -k\,dt + C$

we get $\quad \ln|T-21| = -kt + C$

exponentiate $\quad T - 21 = Ae^{-kt} \qquad (A = e^C)$

and so $\quad T(t) = 21 + Ae^{-kt}$

$T(0) = 5 \Rightarrow \quad 21 + A = 5 \Rightarrow A = -16$

then $\quad T(t) = 21 - 16e^{-kt}$

but $\quad T(1) = 12 \Rightarrow \quad 21 - 16e^{-k} = 12$

$$-16e^{-k} = -9$$

$$k = -\ln(9/16) \simeq 0.5754$$

$$\therefore \quad \boxed{T(t) = 21 - 16e^{-0.5754t}}$$

iii) at noon $\quad T(2) = 21 - 16\,e^{-0.5754(2)} \simeq \boxed{15.9\,°C}$

Ⓐ

2. Parks Canada places a herd of 500 caribou on an island in Hudson Bay. They estimate that the carrying capacity of the island is 2000 caribou. Also, the relative growth rate in an unconstrained environment is estimated to be $k = 0.08$ per year. Assuming that the population follows the Logistic Model,
(i) write the differential equation that the population $P(t)$ will satisfy, where t is measured in years and
(ii) given that the solution of the Logistic equation is $P(t) = \dfrac{M}{1 + Ae^{-kt}}$, find the number of caribou on the island after 2 years.

i,

$$\boxed{\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) = 0.08\,P\left(1 - \frac{P}{2000}\right)}$$

ii,

$$P(t) = \frac{M}{1+Ae^{-kt}} = \frac{2000}{1+Ae^{-0.08t}}$$

$$P(0) = 500 \Rightarrow \frac{2000}{1+A} = 500 \Rightarrow A = 3$$

So $$\boxed{P(t) = \frac{2000}{1+3e^{-0.08t}}}$$

thus $$P(2) = \frac{2000}{1+3e^{-0.08(2)}} = \boxed{562}$$

3.

(i) Determine if the series $\sum_{n=1}^{\infty} \frac{3+\cos(n+1)}{n^3+2010n}$ converges or diverges. Explain your reasoning.

FROM $-1 \le \cos(n+1) \le 1 \Rightarrow 3+\cos(n+1) \le 3+1=4$. Hence:

$$\frac{3+\cos(n+1)}{n^3+2010n} \le \frac{4}{n^3+2010n} \le \frac{4}{n^3}.$$ Hence:

$$\sum_{n=1}^{\infty} \frac{3+\cos(n+1)}{n^3+2010n} \le \sum_{n=1}^{\infty} \frac{4}{n^3} = 4\sum_{n=1}^{\infty} \frac{1}{n^3}$$ WHICH converges (p-series: $p=3>1$)

By Comparison Theorem: $\sum_{n=1}^{\infty} \frac{3+\cos(n+1)}{n^3+2010n}$ is convergent.

(ii) Give an example of a series that is convergent, but not absolutely.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$ etc

(iii) Give an example of power series with an infinite radius of convergence.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

or

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$ or

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

4. Determine the radius and interval of convergence of $\sum_{n=0}^{\infty} \frac{(x+2)^n}{9^n(n+3)}$.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(x+2)^{n+1}}{9^{n+1}(n+4)} \cdot \frac{9^n(n+3)}{(x+2)^n} \right| =$$

$$\lim_{n\to\infty} \frac{|x+2|}{9} \frac{n+3}{n+4} = \lim_{n\to\infty} \frac{|x+2|}{9} \cdot \frac{1+\frac{3}{n}}{1+\frac{4}{n}} = \frac{|x+2|}{9}$$

(iF) $\frac{|x+2|}{9} < 1 \Rightarrow$ my series is convergent.

(iF) $|x+2| < 9 \Rightarrow$ my series is convergent. Hence

$\boxed{R = 9}$. THIS can be written as: $-9 < x+2 < 9$, i.e.

$-11 < x < 7$.

End points: (iF) $\boxed{x = -11}$; my series is: $\sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{n+3}$

By the Alternating Series Test $\Rightarrow$ it converges.

(iF) $x = 7$; my series is: $\sum_{n=0}^{\infty} \frac{1}{n+3}$. it

diverges: $\frac{1}{n+3} \geq \frac{1}{2n}$ for $n \geq 3$. But $\sum_{n=0}^{\infty} \frac{1}{2n}$ is

divergent: p-series with $p = 1$.

CONCLUSION: the interval is $[-11, 7)$ or

$-11 \leq x < 7$

5.

(i) Use the Maclaurin series of $\frac{1}{1-x}$ to find the Maclaurin series of $\frac{1}{1+x^3}$ and give its radius of convergence.

(ii) Then deduce the Maclaurin series of $\int \frac{1}{1+x^3}\,dx$ and give its radius of convergence.

(iii) Find a power series representation of $f(x) = \frac{-1}{(1+x)^2}$.

(i) Recall $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$. So: $\frac{1}{1+x^3} = \frac{1}{1-(-x^3)}$

$= \sum_{n=0}^{\infty} (-x^3)^n = \sum_{n=0}^{\infty} (-1)^n x^{3n}$. Since $R=1$ for

$\sum_{n=0}^{\infty} x^n$, it follows that $\sum_{n=0}^{\infty} (-1)^n x^{3n}$ is 1

(ii) $\int \frac{1}{1+x^3}\,dx \overset{(i)}{=} \int \sum_{n=0}^{\infty} (-1)^n x^{3n}\,dx = C + \sum_{n=0}^{\infty} \int (-1)^n x^{3n}\,dx$

$= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+1}}{3n+1}$; $R=1$ (because of (i))

(iii) $f(x) = -(1+x)^{-2} = \left[(1+x)^{-1}\right]' = \left(\sum_{n=0}^{\infty} (-1)^n x^n\right)'$

$= \boxed{\sum_{n=1}^{\infty} (-1)^n n x^{n-1}}$

B

1. A sandwich is taken out of the refrigerator, which has a constant temperature of 4 °C, at 10:00 and left on the counter. The temperature in the kitchen is constant at 22 °C. One hour later, the temperature of the sandwich is 11 °C. Let $T(t)$ be the temperature of the sandwich t hours after 10:00.

(i) If the temperature of the sandwich follows Newton's Law of Heating, what is the differential equation that $T(t)$ satisfies?

(ii) Solve the differential equation and give an explicit formula for $T(t)$.

(iii) What is the temperature of the sandwich at noon?

i, $$\boxed{\frac{dT}{dt} = -k(T-22)}$$

ii, $$T(t) = 22 + Ae^{-kt}$$

$$T(0) = 4 \Rightarrow 22 + A = 4 \Rightarrow A = -18$$

$$T(t) = 22 - 18e^{-kt}$$

$$T(1) = 11 \Rightarrow 22 - 18e^{-k} = 11$$

$$-18e^{-k} = -11$$

$$k = -\ln(11/18) \simeq 0.4925$$

$$\therefore \boxed{T(t) = 22 - 18e^{-0.4925t}}$$

iii, at noon $$T(2) = 22 - 18e^{-0.4925(2)} \simeq \boxed{15.3\,°C}$$

B

2. Parks Canada places a herd of 400 caribou on an island in Hudson Bay. They estimate that the carrying capacity of the island is 2000 caribou. Also, the relative growth rate in an unconstrained environment is estimated to be $k = 0.10$ per year. Assuming that the population follows the Logistic Model,
(i) write the differential equation that the population $P(t)$ will satisfy, where t is measured in years and
(ii) given that the solution of the Logistic equation is $P(t) = \dfrac{M}{1 + Ae^{-kt}}$, find the number of caribou on the island after 3 years.

i, $$\boxed{\frac{dP}{dt} = 0.10\,P\left(1 - \frac{P}{2000}\right)}$$

ii, $$P(t) = \frac{2000}{1 + Ae^{-0.1t}}$$

$$P(0) = 400 \Rightarrow \frac{2000}{1+A} = 400 \Rightarrow A = 4$$

$$\therefore \boxed{P(t) = \frac{2000}{1+4e^{-0.1t}}}$$

then $$P(3) = \frac{2000}{1+4e^{-0.1(3)}} = \boxed{505}$$

3.

(i) Determine if the series $\sum_{n=1}^{\infty} \frac{2+\sin(n+2010)}{n^4+4n}$ converges or diverges. Explain your reasoning.

Recall that $\sin(n+2010) \le 1$, so $2+\sin(n+2010) \le 2+1=3$.

Hence $\frac{2+\sin(n+2010)}{n^4+4n} \le \frac{3}{n^4+4n} \le \frac{3}{n^4}$.

Hence $\sum_{n=1}^{\infty} \frac{2+\sin(n+2010)}{n^4+4n} \le \sum_{n=1}^{\infty} \frac{3}{n^4} = 3\sum_{n=1}^{\infty} \frac{1}{n^4}$ which converges as a p-series with $p=4>1$

By Comparison Test, we get that $\sum_{n=1}^{\infty} \frac{2+\sin(2010+n)}{n^4+4n}$ is convergent.

(ii) Give an example of power series with an infinite radius of convergence.

$\sum_{n=0}^{\infty} \frac{x^n}{n!}$ or $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ or $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$.

(iii) Give an example of a series that is absolutely convergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$$

4. Determine the radius and interval of convergence of $\sum_{n=0}^{\infty} \frac{(x+3)^n}{8^n(n+2)}$.

Note that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(x+3)^{n+1}}{8^{n+1}(n+3)} \cdot \frac{8^n(n+2)}{(x+3)^n} \right|$

$= \lim_{n\to\infty} \frac{|x+3|}{8} \frac{n+2}{n+3} = \lim_{n\to\infty} \frac{|x+3|}{8} \cdot \frac{1+2/n}{1+3/n} = \frac{|x+3|}{8}$.

If $\frac{|x+3|}{8} < 1$, then my series is convergent. So:

If $|x+3| < 8$, my series is convergent. Hence we may write $-8 < x+3 < 8$, or $-11 < x < 5$

$R = 8$.

End points: If $\boxed{x=-11}$, my series becomes:

$\sum_{n=0}^{\infty} (-1)^n \frac{1}{n+2}$ which converges by Alternating Series Test

If $x = 5$, then my series becomes: $\sum_{n=0}^{\infty} \frac{1}{n+2}$.

Note $\frac{1}{n+2} \geq \frac{1}{2n}$ (for $n \geq 2$). So

$\sum \frac{1}{n+2} \geq \sum \frac{1}{2n} = \frac{1}{2} \sum \frac{1}{n}$ which diverges as a p-series: $\boxed{p=1}$. By Comparison test $\Rightarrow$ my series is Divergent.

Conclusion: the interval is $[-11, 5)$ or

$-11 \leq x < 5$

5.

(i) Use the Maclaurin series of $\dfrac{1}{1-x}$ to find the Maclaurin series of $\dfrac{1}{1+x^4}$ and give its radius of convergence.

(ii) Then deduce the Maclaurin series of $\displaystyle\int \frac{1}{1+x^4}\,dx$ and give its radius of convergence.

(iii) Find a power series representation of $f(x) = \frac{3}{(1+x)^2}$.

(i) Recall $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$ with $R=1$. Hence

$$\frac{1}{1+x^4} = \frac{1}{1-(-x^4)} = \sum_{n=0}^{\infty} (-x^4)^n = \sum_{n=0}^{\infty} (-1)^n x^{4n}.$$ The

last series has $\underline{R=1}$.

(ii) $$\int \frac{1}{1+x^4}\,dx = C + \sum_{n=0}^{\infty} \int (-1)^n x^{4n}\,dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{4n+1}$$

Here: $\underline{R=1}$ (see (i))

(iii) $$f(x) = 3(1+x)^{-2} = -3\left[(1+x)^{-1}\right]' = -3\left[\sum_{n=0}^{\infty} (-1)^n x^n\right]'$$

$$= -3\sum_{n=1}^{\infty} (-1)^n n\, x^{n-1} = \sum_{n=1}^{\infty} 3(-1)^n n\, x^{n-1}$$