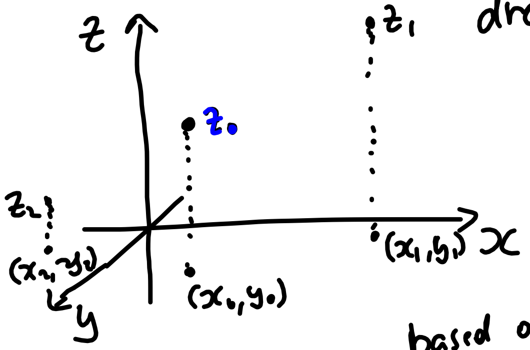
Functions in 2 variables (§14.1)

recall: $y = f(x)$ function in x, one variable.

NOW: $z = f(x,y)$ function in x, y : two variables.

drawing graph: now in 3 dimensions

draw $z = f(x,y)$
pick (x_0, y_0): compute $z_0 = f(x_0, y_0)$

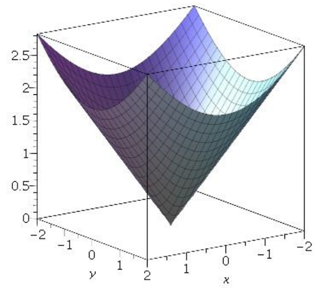


(x,y)-plane

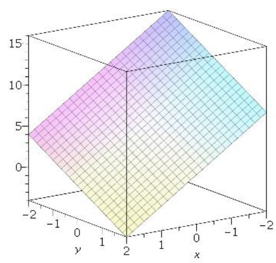
based on the 'height' (z-value), try to guess the surface above (x,y)-plane.

Use cross section curves from last week, to 'guess' better.

Ex 1 $f(x,y) = \sqrt{x^2 - y^2}$

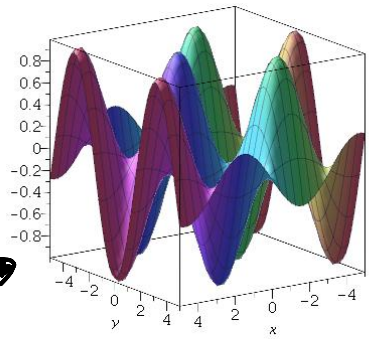


Ex 2: $f(x,y) = 6 - 3x - 2y$



a plane

Ex 3: $f(x,y) = \sin(x) \cdot \cos(y)$

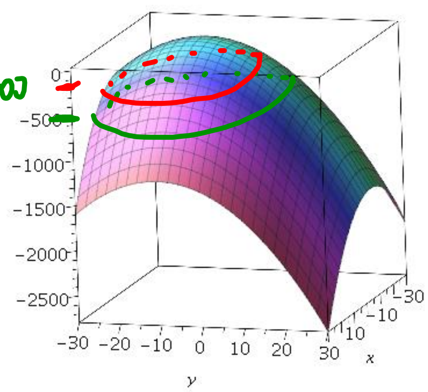


Again: talk about domain D:

domain $D \subseteq \mathbb{R} \times \mathbb{R}$ (region in (x,y)-plane) for which f is defined.

Level curves: those are curves on the graph of a function $z=f(x,y)$, where every point has the same z-value (compare: altitude lines)

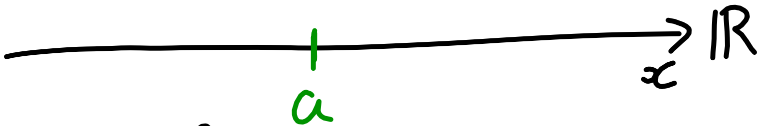
fix $z=-500$



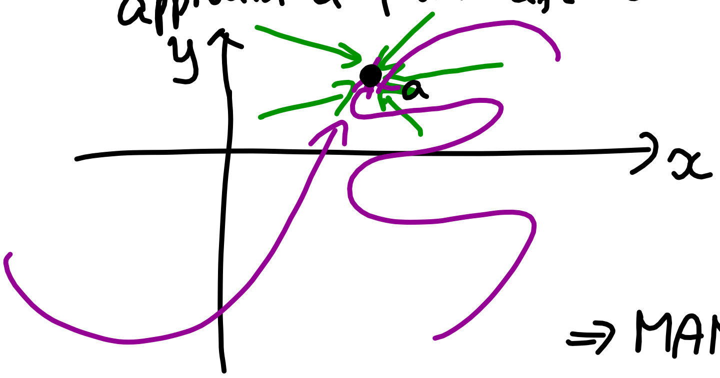
} level curves.

Limits & Continuity (§14.2)

recall: limit for $y=f(x)$: $\lim_{x\to a} f(x)$



approach a from left or right.



(x,y)-plane

Can approach a in a straight line

or: approach curved:

$\Rightarrow$ MANY ways to approach a.

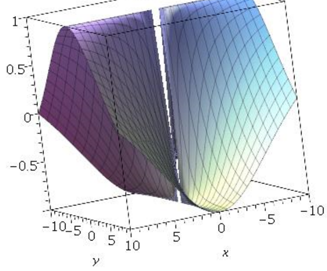
Ex $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ domain: $\mathbb{R}^2 \setminus \{0,0\}$

'entire (x,y)-plane without the point $(0,0)$'

Case I: approach along the y-axis:

set $x=0$:

$$\lim_{(0,y)\to(0,0)} \frac{0^2-y^2}{0^2+y^2} = \lim_{(0,y)\to(0,0)} \frac{-\cancel{y^2}}{\cancel{y^2}} = \underline{\underline{-1}}$$



Case II: along x-axis, set $y=0$

$$\lim_{(x,0)\to(0,0)} \frac{x^2-0^2}{x^2+0^2} = \lim_{(x,0)\to(0,0)} \frac{\cancel{x^2}}{\cancel{x^2}} = \underline{\underline{+1}}$$

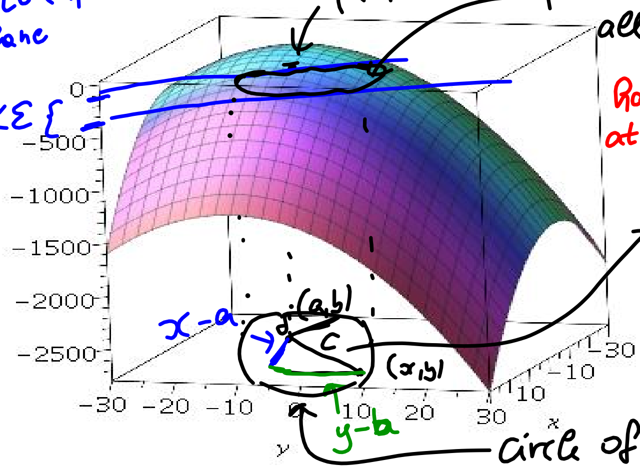
We get two different values for the limit by approaching it from two different directions, so the limit does not exist (DNE). Compare CALC I: left limit is not equal to the right limit: limit DNE.

Def (Limit): The limit of $f(x,y)$ as (x,y) approaches (a,b) is $L = \displaystyle\lim_{(x,y)\to(a,b)} f(x,y)$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if (x,y) is in the domain of f and

$$0 < \sqrt{(x-a)^2+(y-b)^2} < \delta \quad \text{then}$$

$$|f(x,y) - L| < \varepsilon.$$

distance of (x,y) to (a,b) in plane



$f(a,b)$

function values of all points in circle: have to differ at most ε

$<\varepsilon$

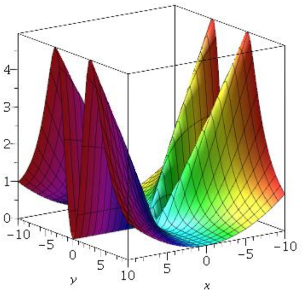
Pythagoras: $(x-a)^2+(y-b)^2 = c^2$

$x-a$ (a,b) c (x,y) $y-b$

circle of radius δ

<u>Ex</u> $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^2+y^4}$

First, guess the limit: take small x, y values, put into function. eg: $x = 0.05$, $y = 0.05$ ...



$\rightsquigarrow$ 'guess' limit is zero.
(no proof, just to get an idea)

No marks for just putting in values.

(1) assume $\sqrt{(x-0)^2+(y-0)^2} < \delta$
(2) get an equation in δ and ε from $|f(x,y) - 0| < \varepsilon$
(3) express δ in terms of ε (as a function of ε)

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(1) $\sqrt{x^2+y^2} < \delta$

(2) $$\left| \frac{x^2y^2}{x^2+y^4} - 0 \right| < \varepsilon$$

$$\left| \frac{x^2y^2}{x^2+y^4} \right| < \varepsilon$$

$y^4 > 0$ so $x^2 + y^4 \geq x^2$

if $$\left| \frac{x^2y^2}{x^2+y^4} \right| \leq \frac{\cancel{x^2}y^2}{\cancel{x^2}} < \varepsilon$$

$$y^2 < \varepsilon$$

in particular as $\sqrt{x^2+y^2} < \delta$

need to set δ such that $y^2 < \varepsilon$.

if $\sqrt{x^2+y^2} < \delta$ and we need $y^2 < \varepsilon$

$$x^2 + y^2 < \delta^2$$

if we say $\underset{\geq 0}{x^2} + y^2 < \varepsilon$ then also $y^2 < \varepsilon$.

So set $\delta^2 = \varepsilon$ to assure $y^2 < \varepsilon$.

$\boxed{\delta = \sqrt{\varepsilon}}$ (an upper bound)

Ex2 $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4-y^4}{x^2+y^2}$

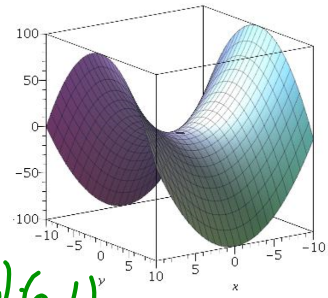
again: putting in small values suggest limit is 0.

(1) assume $\sqrt{x^2+y^2} < \delta$

(2)

$|f(x,y) - 0| < \varepsilon$

$\left|\dfrac{x^4-y^4}{x^2+y^2}\right| < \varepsilon$



$\boxed{\text{use: } a^2-b^2 = (a+b)(a-b)}$

So: $x^4 - y^4 = (x^2+y^2)(x^2-y^2)$

$\left|\dfrac{\cancel{(x^2+y^2)}(x^2-y^2)}{\cancel{x^2+y^2}}\right| < \varepsilon$

$|x^2-y^2| < \varepsilon$

satisfy this, based on $\sqrt{x^2+y^2} < \delta$.

$|a-b| \le |a| + |b|$ abs. value rule

can make it bigger and see then if it is still satisfied.

using this, $|\underbrace{x^2}_{\ge 0}| + |y^2| < \varepsilon$

$x^2 + y^2 < \varepsilon$

$\delta^2 = (x^2+y^2) < \varepsilon$

So $\delta^2 < \varepsilon$

$\Rightarrow \delta < \sqrt{\varepsilon}$

Def A function f of 2 variables is called **continuous at (a,b)** if $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

Continuous on $D \subseteq \mathbb{R}^2$: continuous at all $(x,y) \in D$ (all points in region)

D: domain, or any subset of plane

Ex $f(x,y) = \dfrac{x^2y^2}{x^4+3y^4}$

if type in $x=0.05$, $y=0$: limit suggests 0.

Case I: approach along $y=0$ (x-axis):

$$\lim_{(x,0)\to(0,0)} \frac{x^2\cdot 0^2}{x^4+3\cdot 0^4} = 0$$

case II: along y-axis, so $x=0$:

$$\lim_{(0,y)\to(0,0)} \frac{0^2\cdot y^2}{0^4+3\cdot y^4} = 0$$

not enough for anything

does not show $\lim = 0$.

BUT: approach along $y=x$

$$\lim_{(x,x)\to(0,0)} \frac{x^2\cdot x^2}{x^4+3x^4} = \lim_{(x,x)\to(0,0)} \frac{\cancel{x^4}}{4\cdot \cancel{x^4}} = \frac{1}{4}$$

$\Rightarrow$ So limit DNE