UNIVERSITY OF CALGARY

UNIVERSITY OF CALGARY
FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS AND STATISTICS
MIDTERM EXAMINATION 1
AMAT 311, Lec 01 - Fall, 2015

DATE: 29/October/2015 **Time: 75 minutes**

Student ID Number:	Last Name:	First Name:

EXAMINATION RULES

1. This is a closed book examination.
2. A one-page self-prepared formula sheet is allowed for this examination.
3. For questions 1-10, circle the correct answer; for questions 11-13, write your work in detail at the provided space.
4. The use of personal electronic or communication devices is prohibited.
5. A University of Calgary Student ID card is required to write the Final Examination and could be requested for midterm examinations. If adequate ID isn't present the student must complete an Identification Form.
6. Students late in arriving will not be permitted after one-half hour of the examination time has passed.
7. No student will be permitted to leave the examination room during the first 30 minutes, nor during the last 15 minutes of the examination. Students must stop writing and hand in their exam immediately when time expires.
8. All inquiries and requests must be addressed to the exam supervisor.
9. Students are strictly cautioned against:
    a. communicating to other students;
    b. leaving answer papers exposed to view;
    c. attempting to read other students' examination papers
10. During the final examination, if a student becomes ill or receives word of domestic affliction, the student must report to the Invigilator, hand in the unfinished paper and request that it be cancelled. If ill, the student must report immediately to a physician/counselor for a medical note to support a deferred examination application.
11. Once the examination has been handed in for marking, a student cannot request that the examination be cancelled. Retroactive withdrawals from the course will be denied.
12. Failure to comply with these regulations will result in rejection of the examination paper.

Question	Total Marks	Actual Marks
1-5	20	
6	7	
7	7	
8	8	
9	8	
10	8	
11	14	
12	14	
13	14	
Total	100	

Instructions: For questions 1-10, circle the correct answer, show some work if necessary. For questions 11-13, show complete solutions.

1. **(4 marks)** (**True** or **False**) The function $y = xe^{-x}\sin(3x)$ is a solution of $ay'' + by' + cy = 0$, where a, b and c are constants, and $a \neq 0$.

   (A) True   **(B)** False

   $x \cdot e^{-x} \cdot \sin(3x) \longrightarrow r = -1 \pm 3i$ are repeated roots of $ar^2 + br + c = 0$. However this is impossible as $ar^2 + br + c = 0$ can have at most two roots

2. **(4 marks)** The differential equation $y'' + x^2 y = \cos x$ is ________

   (A) Second order, linear   **(B)** Second order, nonlinear

   (C) First order, linear   **(D)** First order, nonlinear

   (E) None of the above

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3. **(4 marks)** The Wronskian of the set of functions $\{t, \sin t, \cos t\}$ is ________

(A) $\sin t + \cos t$ (B) $t \sin t$

(C) $t \cos t$ (D) $-t$

(E) $t+1$

$$W = \begin{vmatrix} t & \sin t & \cos t \\ 1 & \cos t & -\sin t \\ 0 & -\sin t & -\cos t \end{vmatrix} = t \cdot \begin{vmatrix} \cos t & -\sin t \\ -\sin t & -\cos t \end{vmatrix} - 1 \cdot \begin{vmatrix} \sin t & \cos t \\ -\sin t & -\cos t \end{vmatrix} = -t.$$

4. **(4 marks)** If $e^{2x} \cos x$ is a solution of the differential equation

$$y'' + by' + cy = 0,$$

where b and c are constants, then

(A) $b = 2, \quad c = 1$ (B) $b = 4, \quad c = -5$

(C) $b = -4, \quad c = 5$ (D) $b = 3, \quad c = 6$

(E) $b = -2, \quad c = -1$

The solution is $e^{2x} \cdot \cos x \Rightarrow r_1 = 2+i, \quad r_2 = 2-i$. So the char. eq is

$[r-(2+i)] \cdot [r-(2-i)] = r^2 - 4r + 5 = 0 \Rightarrow$

$b = -4, \; c = 5.$

5. **(4 marks)** Consider the initial-value problem

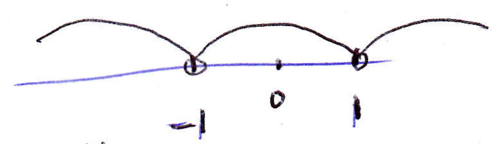
$$(x^2+2)y'' + \frac{x}{x+1}y' + y = \frac{\sin x}{x-1}, \quad y(0) = 3.$$

The **largest** interval on which a unique solution is **guaranteed** to exist is

(A) $(-\infty, \quad -1)$ (B) $(-1, \quad 0)$

(C) $(-1, \quad 1)$ (D) $(0, \quad \infty)$

(E) $(1, \quad \infty)$



The Eq. can be rewritten

as $y'' + \frac{x}{(x^2+2)(x+1)}y' + \frac{1}{x^2+2}y = \frac{\sin x}{(x^2+2)(x-1)}$.

The discts points are $x=-1$, $x=1$ shown above.

Only $(-1, 1)$ contains the point $x_0 = 0$.

6. **(7 marks)** The mass (in grams) of a radioactive substance is given by

$$Q(t) = 3.9e^{-\frac{\ln 2}{10}t},$$

where the unit of time t is day, the half-life of the radioactive substance is

(A) $\tau = 10$ *days* (B) $\tau = 7.5$ *days*

(C) $\tau = 3.9$ *days* (D) $\tau = 2$ *days*

(E) $\tau = \ln 2$ *days*

$k \cdot \tau = -\ln 2$,

From the solution

$Q(t) = 3.9\, e^{-\frac{\ln 2}{10}t}$ $\rightarrow$ $k = -\frac{\ln 2}{10}$.

$\Rightarrow \tau = \frac{-\ln 2}{k}$

$\Rightarrow \tau = 10$

$r^2+r-2=0$ $\quad r=1,\ r=-2$

$(r+2)(r-1)=0$.

7. **(7 marks)** Let y_p be a particular solution of the differential equation:

$$y'' + y' - 2y = (13 - x)e^{x} - e^{-2x}\sin(x),$$

then y_p is in the form of ($A_0, A_1, B_0, B_1, C_0, C_1$ are constants)__________ .

(A) $e^{x}(A_0x + A_1x^2) + e^{-2x}[(B_0x + B_1x^2)\cos(x) + (C_0x + C_1x^2)\sin(x)]$

(B) $e^{x}(A_0x + A_1x^2) + e^{-2x}[B_0\cos(x) + C_0\sin(x)]$

(C) $e^{x}(A_0x + A_1x^2) + e^{-2x}[(B_0 + B_1x)\cos(x) + (C_0 + C_1x)\sin(x)]$

(D) $e^{x}(A_0 + A_1x) + e^{-2x}[(B_0x + B_1x^2)\cos(x) + (C_0x + C_1x^2)\sin(x)]$

(E) $e^{x}(A_0 + A_1x) + e^{-2x}[B_0\cos(x) + C_0\sin(x)]$

$r^2+r-2 \Rightarrow r_1=1,\ r_2=-2.$

$(13-x)e^x \longrightarrow e^x\cdot x^1\cdot[A_1x+A_0] = e^x[A_0x+A_1x^2].$

$-e^{-2x}\sin x \longrightarrow e^{-2x}[B_0\cos x + C_0\sin x].$

8. **(8 marks)** The following differential equation is **exact**,

$xy^3+g(x)$

$y^3+g' = 2x+y^3$

$g=x^2$

$$(2x + y^3)dx + 3xy^2dy = 0,$$

which of the following is its solution (C is a constant):

(A) $3xy^2 + xy = C$ **(B)** $x^2 + y^3 = C$

(C) $x^2 + xy^3 + y = C$ **(D)** $x^2 + xy^3 = C$

(E) $x^2 + y^3 = C$

$F(x,y) = \int(2x+y^3)\,dx + g(y) = x^2+xy^3+g(y)$

$\frac{\partial F}{\partial y} = 3xy^2+g'(y)$ let $\frac{\partial F}{\partial y} = N(x,y) \Rightarrow$

$3xy^2+g'(y) = 3xy^2 \Rightarrow g'(y)=0 \Rightarrow g(y)=0.$

then $F(x,y) = x^2+xy^3.$

9. **(8 marks)** An object with initial temperature of 100°C is placed outside on a day when the temperature is −20°C. The temperature of the object becomes 20°C in 4 minutes, about the temperature $T(t)$ of the object at time t, which of the following is correct?

(A) $T(t) = -20 + 120e^{-\frac{\ln 3}{4}t}$

(B) $T(t) = 100 + 120e^{\frac{\ln 3}{4}t}$

(C) $T(t) = 20 + 80e^{-\frac{\ln 3}{4}t}$

(D) $T(t) = 20 + 80e^{-\frac{3}{4}t}$

(E) $T(t) = -20 + 120e^{-\frac{3}{4}t}$

$T' = -k(-20 - T) = k(20 + T).$

$\Rightarrow T = -20 + ce^{kt}$

$T(0) = 100 \Rightarrow 100 = -20 + c \Rightarrow c = 120$

So $T(t) = -20 + 120e^{kt}$

$T(4) = 20 \Rightarrow -20 + 120e^{k\cdot 4} = 20 \Rightarrow e^{k\cdot 4} = \frac{40}{120} = \frac{1}{3} \Rightarrow 4\cdot k = \ln\frac{1}{3}$

$\Rightarrow k = -\frac{\ln 3}{4}$. So

$T(t) = -20 + 120e^{-\frac{\ln 3}{4}\cdot t}$

10. **(8 marks)** The general solution of

$$y'' + 4y' + 20y = 0$$

is ____________ . (c_1 and c_2 are arbitrary constants.)

(A) $c_1e^{-4t} + c_2te^{-5t}$

(B) $c_1e^{-4t} + c_2e^{-5t}$

(C) $c_1e^{-2t}\cos(4t) + c_2e^{-2t}\sin(4t)$

(D) $c_1e^{-2t}\cos(2t) + c_2e^{-2t}\sin(2t)$

(E) $c_1\cos(4t) + c_2e^{-2t}\sin(4t)$

$r^2 + 4r + 20 = 0 \Rightarrow (r+2)^2 + 16 = 0 \Rightarrow$

$r_1 = -2 + 4i, \quad r_2 = -2 - 4i.$

so $y(t) = c_1 e^{-2t}\cdot\cos(4t) + c_2\cdot e^{-2t}\cdot\sin(4t)$

11. **(14 marks)** Solve the initial-value problem of the Bernoulli equation (Assume $t > 0$):

$$t^2y' + 2ty - y^3 = 0, \quad y(1) = 1.$$

Rewrite the DE: $y' + \frac{2}{t}y = t^{-2}\cdot y^3$.

so $f(t) = t^{-2}$, $n = 3$.

let $v = y^{1-3} = y^{-2} \Rightarrow v' = -2\cdot y^{-3}\cdot y' \Rightarrow y' = -\frac{1}{2}y^3\cdot v'$

then $-\frac{1}{2}y^3\cdot v' + \frac{2}{t}y = t^{-2}\cdot y^3$.

so $-\frac{1}{2}v' + \frac{2}{t}y^{-2} = t^{-2}. \Rightarrow -\frac{1}{2}v' + \frac{2}{t}v = t^{-2}$

$\Rightarrow v' - \frac{4}{t}v = -2t^{-2}$.

solve for v: $(t^{-4}\cdot v)' = -2t^{-6} \Rightarrow$

$t^{-4}\cdot v = \frac{2}{5}t^{-5} + c \Rightarrow v = \frac{2}{5}t^{-1} + ct^4$.

$y(t) = \left(\frac{1}{v}\right)^{\frac{1}{2}} = \sqrt{\frac{1}{\frac{2}{5}t^{-1} + ct^4}} = \frac{1}{\sqrt{\frac{2}{5}t^{-1} + ct^4}}$.

$y(1) = 1 \Rightarrow \frac{1}{\sqrt{\frac{2}{5} + c}} = 1 \Rightarrow 1 = \frac{1}{\frac{2}{5} + c} \Rightarrow$

$c = \frac{3}{5}$. So $y(t) = \frac{1}{\sqrt{\frac{2}{5}t^{-1} + \frac{3}{5}t^4}}$

12. **(14 marks)** A tank initially contains 400 litres of a salt solution with a concentration of 0.25 g/litre. A solution with a salt concentration of 2.5 g/litre is added to the tank at 4 litres/minute, and the resulting mixture is drained out at 4 litres/minute.

(a) Find the amount $Q(t)$ of the salt in the container at time t.

(b) Find the limit of $Q(t)$ when $t \to +\infty$.

$$Q' = r_{in} \cdot C_{in} - r_{out} \cdot C_{out}$$

$$\Rightarrow Q' = 4 * 2.5 - 4 \cdot \frac{Q}{400}, \qquad Q(0) = 100$$

then $Q' = -\frac{Q}{100} + 10$, so

$$Q' + \frac{1}{100} Q = 10.$$

then $e^{\frac{t}{100}} \cdot Q = \int 10 \cdot e^{\frac{t}{100}} \, dt$

so $e^{\frac{t}{100}} \cdot Q = 1000 e^{\frac{t}{100}} + C.$

then $Q(t) = 1000 + Ce^{-\frac{t}{100}}.$

$Q(0) = 100 \Rightarrow C = -900$

then $Q(t) = 1000 - 900 e^{-\frac{t}{100}}.$

$$\lim_{t \to \infty} Q(t) = 1000.$$

13. **(14 marks)** Find the **general solution** of the following differential equation

$$y'' - 2y' + y = \frac{e^t}{t^2}.$$

First. Find y_1 and y_2 for $y'' - 2y' + y = 0$. $r^2 - 2r + 1 = 0$.
$\Rightarrow r_1 = r_2 = 1$

$\Rightarrow y_1 = e^t, \quad y_2 = te^t.$

then $W(t) = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t}.$

let $y_p = u_1 e^t + u_2 \cdot te^t$ with

$$u_1 = -\int \frac{te^t \cdot \frac{e^t}{t^2}}{e^{2t}} dt = -\int \frac{1}{t} dt = -\ln t$$

$$u_2 = \int \frac{e^t \cdot \frac{e^t}{t^2}}{e^{2t}} dt = \int \frac{1}{t^2} dt = -\frac{1}{t}$$

then $y_p = -e^t \cdot \ln t - e^t.$

and $y(t) = c_1 e^t + c_2 te^t - e^t \ln t - e^t$

THE END