UNIVERSITY OF TORONTO
Joseph L. Rotman School of Management

Oct. 25, 2016 | | **Burke/Corhay/Kan**
RSM332 | **MID-TERM EXAMINATION** | **Geoffrey/Wang**

SOLUTIONS

1. (a) Since the mortgage interest rate in Canada is a semi-annually compounded rate, the monthly interest rate is given by

$$r_m = \left(1 + \frac{0.04}{2}\right)^{\frac{1}{6}} - 1 = 0.003306.$$

We plan to repay the mortgage using $30 \times 12 = 360$ monthly payments. Therefore, the monthly mortgage payment is

$$C = \frac{500000}{A_{r_m}^{360}} = \frac{500000 r_m}{1 - \frac{1}{(1+r_m)^{360}}} = \$2377.59.$$

(b) Let r^* be the bi-monthly interest rate, we have

$$r^* = (1 + r_m)^2 - 1 = 0.006623.$$

You plan to repay the mortgage using $30 \times 6 = 180$ bi-monthly payments. Therefore, the bi-monthly mortgage payment is

$$C^* = \frac{500000}{A_{r^*}^{180}} = \frac{500000 r^*}{1 - \frac{1}{(1+r^*)^{180}}} = \$4763.05.$$

(c) After you make the 9th bi-monthly payment, you owe the bank $180 - 9 = 171$ bi-monthly payments, so the outstanding balance is $C^* A_{r^*}^{171}$. It follows that the interest portion for the 10th bi-monthly payment is

$$C^* A_{r^*}^{171} r^* = C^* \left[1 - \frac{1}{(1 + r^*)^{171}}\right] = \$3222.50,$$

and the principal payment portion for the 10th bi-monthly payment is $\$4763.05 - \$3222.50 = \$1540.55$.

(d) The remaining mortgage balance at the end of 4 years is

$$C^* A_{r^*}^{180-4\times 6} = \frac{C^*}{r^*}\left[1 - \frac{1}{(1+r^*)^{156}}\right] = \$462372.41.$$

Therefore, the maximum amount that you can borrow is

$$0.8 \times \$650000 - \$462372.41 = \$57627.59.$$

(e) The monthly interest rate for the new mortgage is

$$r_m = \left(1 + \frac{0.05}{2}\right)^{\frac{1}{6}} - 1 = 0.004124.$$

Since your mortgage payments form a growing annuity, their present value is

$$\begin{aligned}
\text{PV} &= \frac{C_1}{r_m - g}\left[1 - \left(\frac{1+g}{1+r_m}\right)^{10\times 12}\right] \\
&= \frac{C_1}{0.004124 - 0.005}\left[1 - \left(\frac{1.005}{1.004124}\right)^{120}\right] \\
&= 125.93 C_1,
\end{aligned}$$

where C_1 is the first monthly payment. Setting the present value equal to \$50000, we obtain $C_1 = \$50000/125.93 = \$397.05$.

2. (a) Your consumption today and tomorrow are given by

$$\begin{aligned}
C_0 &= Y_0 - I_0 = 1200 - I_0, \\
C_1 &= Y_1 + 40 I_0^{\frac{1}{2}} = 200 + 40 I_0^{\frac{1}{2}}.
\end{aligned}$$

Substituting these two equations into the utility function, we obtain

$$U(C_0, C_1) = (1200 - I_0)^{\frac{1}{2}} (200 + 40 I_0^{\frac{1}{2}})^{\frac{1}{2}}.$$

Differentiating U with respect to I_0, we obtain

$$\frac{\mathrm{d}U}{\mathrm{d}I_0} = -\frac{(200 + 40 I_0^{\frac{1}{2}})^{\frac{1}{2}}}{2(1200 - I_0)^{\frac{1}{2}}} + \frac{(1200 - I_0)^{\frac{1}{2}}}{2(200 + 40 I_0^{\frac{1}{2}})^{\frac{1}{2}}} \frac{40}{2 I_0^{\frac{1}{2}}}.$$

Setting it equal to zero, we obtain

$$\begin{aligned}
I_0^{\frac{1}{2}}(200 + 40 I_0^{\frac{1}{2}}) &= 20(1200 - I_0) \\
\Rightarrow \quad 0 &= 60 I_0 + 200\sqrt{I_0} - 24000 \\
\Rightarrow \quad \sqrt{I_0} &= \frac{-200 + \sqrt{(200)^2 + 4 \times 60 \times 24000}}{2 \times 60} \\
\Rightarrow \quad \sqrt{I_0} &= 18.4 \\
\Rightarrow \quad I_0 &= 338.66.
\end{aligned}$$

(b) The optimal investment is obtained by setting

$$\begin{aligned} \text{MRT} &= -(1+r) \\ \Rightarrow \quad -\frac{\mathrm{d}40\sqrt{I_0}}{\mathrm{d}I_0} &= -1.1 \\ \Rightarrow \quad -\frac{20}{\sqrt{I_0}} &- 1.1 \\ \Rightarrow \quad \sqrt{I_0} &= \frac{20}{1.1} \\ \Rightarrow \quad I_0 &= 330.58. \end{aligned}$$

Therefore, the optimal investment is $I_0^* = 330.58$ and the NPV of the farm is

$$\text{NPV} = -I_0^* + \frac{40\sqrt{I_0^*}}{1+r} = -330.58 + \frac{40\sqrt{330.58}}{1.1} = 330.58.$$

This is the maximum amount that you are willing to pay for the farm.

(c) After you pay the maximum amount for the farm, then the farm (after the payment) will have zero NPV to you. As a result, the optimal consumption is obtained by solving the following maximization problem

$$\begin{aligned} &\max_{C_0, C_1} C_0^{\frac{1}{2}} C_1^{\frac{1}{2}} \\ \text{s.t. } \; & C_0 + \frac{C_1}{1.1} = Y_0 + \frac{Y_1}{1.1} = 1200 + \frac{200}{1.1} = 1381.82. \end{aligned}$$

The budget constraint implies

$$C_1 = 1.1(1381.82 - C_0) = 1520 - 1.1C_0.$$

Substituting this into the utility function, we have

$$U(C_0, C_1) = C_0^{\frac{1}{2}}(1520 - 1.1C_0)^{\frac{1}{2}}.$$

Differentiating this with respect to C_0, we obtain

$$\begin{aligned} \frac{\mathrm{d}U}{\mathrm{d}C_0} &= \frac{1}{2}C_0^{-\frac{1}{2}}(1520 - 1.1C_0)^{\frac{1}{2}} + \frac{1}{2}C_0^{\frac{1}{2}}(-1.1)(1520 - 1.1C_0)^{-\frac{1}{2}} \\ &= \frac{C_0^{-\frac{1}{2}}(1520 - 1.1C_0)^{-\frac{1}{2}}}{2}(1520 - 1.1C_0 - 1.1C_0). \end{aligned}$$

Setting it equal to zero, we obtain $C_0^* = 1520/2.2 = 690.91$. It follows that $C_1^* = 1520 - 1.1C_0^* = 760$.

3. (a) The effective interest rate of the investment product from ABC is

$$r_e = \left(1 + \frac{0.0495}{12}\right)^{12} - 1 = 0.05064.$$

The effective interest rate of the investment product from DEF is

$$r_e = e^{0.0490} - 1 = 0.05022.$$

Since the investment product from ABC has a superior return, you should choose ABC.

(b) Let C be the annual payments. Since there are 42 annual payments, the present value is given by

$$CA_{0.05064}^{42} = \frac{C}{0.05064}\left[1 - \frac{1}{(1.05064)^{42}}\right] = 17.2676C.$$

Setting it equal to \$5,000,000, we get

$$C = \frac{\$5{,}000{,}000}{17.2676} = \$289{,}559.14.$$

(c) After the 5th payment, the remaining value of your investment is

$$CA_{0.05064}^{42-5} = \frac{\$289{,}559.14}{0.05064}\left[1 - \frac{1}{(1.05064)^{37}}\right] = \$4{,}798{,}798.09.$$

(d) Let C_1 be the first payment in the restructured investment plan. The present value of the payments from the restructured investment plan is

$$\begin{aligned} \text{PV} &= \frac{C_1}{r-g}\left[1 - \left(\frac{1+g}{1+r}\right)^T\right] \\ &= \frac{C_1}{0.05064 - 0.02}\left[1 - \left(\frac{1.02}{1.05064}\right)^{37}\right] \\ &= 21.72C_1. \end{aligned}$$

Setting it equal to \$4,798,798.09, we get

$$C_1 = \frac{\$4{,}798{,}798.09}{21.72} = \$220{,}938.62.$$

(e) The dollar amount of the last payment is

$$C_{37} = C_1(1+g)^{36} = \$220{,}938.62(1.02)^{36} = \$450{,}689.89.$$

4. (a) Calculate the spot rates:

$$\begin{aligned}
\mathrm{DF}_1 &= \frac{1}{1+r_1} = \frac{92.59}{100} = 0.9259 &\Rightarrow\quad r_1 = 8.00\%,\\
\mathrm{DF}_2 &= \frac{1}{(1+r_2)^2} = \frac{85.34}{100} = 0.8534 &\Rightarrow\quad r_2 = 8.25\%,\\
\mathrm{DF}_3 &= \frac{1}{(1+r_3)^3} = \frac{782.91}{1000} = 0.78291 &\Rightarrow\quad r_3 = 8.50\%,\\
\mathrm{DF}_4 &= \frac{1}{(1+r_4)^4} = \frac{7149.62}{10000} = 0.714962 &\Rightarrow\quad r_4 = 8.75\%.
\end{aligned}$$

Calculate the forward rates:

$$\begin{aligned}
f_2 &= \frac{\mathrm{DF}_1}{\mathrm{DF}_2} - 1 = 0.0850 = 8.50\%,\\
f_3 &= \frac{\mathrm{DF}_2}{\mathrm{DF}_3} - 1 = 0.0900 = 9.00\%,\\
f_4 &= \frac{\mathrm{DF}_3}{\mathrm{DF}_4} - 1 = 0.0950 = 9.50\%.
\end{aligned}$$

(b) The interest rate of the borrowing/lending contract for the fourth year is 9%, which is different from the theoretical forward rate ($f_4 = 9.50\%$) for the year. So an arbitrage opportunity exists.

To demonstrate an arbitrage opportunity, we "pay low and receive high." We enter the contract today to borrow \$10,000 at the beginning of year 4, which ensures that we will pay interest of 9% for the \$10,000 loan. At the same time, we use a combination of B3 and B4: short-sell 10 units of B3 (receiving \$7,829.1) and buy 1.095 (=7829.1/7149.62) units of B4 (paying \$7,829.1), which effectively ensures that we will receive interest of 9.5% on lending \$10,000 for year 4. The details are in the table below.

	$t = 0$	$t = 3$	$t = 4$
Borrow \$10,000 using the contract		10,000	−10,900
Short-sell 10 units of bond B3	7,829.1	−10,000	
Buy 1.095 units of bond B4	−7,829.1		10,950
Total	0	0	50

This shows that the transactions lead to a positive profit of \$50 at the end of year 4, but no net cash flow at any other time. So this is a risk-free

(c) If $f_2 < 0$, then $\mathrm{DF}_1 < \mathrm{DF}_2$ since

$$\frac{\mathrm{DF}_1}{\mathrm{DF}_2} = 1 + f_2 < 1.$$

To demonstrate an arbitrage opportunity, we "buy low and sell high." We buy a 1-year zero coupon bond and short-sell a 2-year zero coupon bond, and today we receive a positive sum of money, equal to $100 \times (\mathrm{DF}_2 - \mathrm{DF}_1)$. We won't have any problem later on. One year later we receive \$100 (due to the buying of 1-year zero coupon bond) and we keep the cash and hold it to the end of year 2. That will be exactly what we need to pay off the \$100 we owe (due to the short-selling of 2-year zero coupon bond).

So this is a risk-free arbitrage opportunity: a positive income today and no future obligations (no out flow of money) at any points in the future.

5. (a) The price of the three securities satisfy the following system of equations:

$$\begin{aligned}
\frac{1080}{1+0.05} &= \frac{1080}{1+r_1}, \\
\frac{100}{1+0.0643} + \frac{1100}{(1+0.0643)^2} &= \frac{100}{1+r_1} + \frac{1100}{(1+r_2)^2}, \\
\frac{1000}{(1+0.07)^3} &= \frac{1000}{(1+r_3)^3}.
\end{aligned}$$

From equations (1) and (3), we directly obtain $r_1 = 5\%$, and $r_3 = 7\%$. Finally, r_2 is obtained by replacing for r_1 in equation (2), $r_2 = 6.5\%$.

(b) (i) Using the term structure of spot rates calculated in (a), the price of the newly issued bond in the absence of arbitrage is:

$$P_B = \frac{60}{1+0.05} + \frac{60}{(1+0.065)^2} + \frac{1060}{(1+0.07)^3} = \$975.32$$

(ii) The traded price of the bond (\$950) differs from that implied by the other bonds (\$975.32), so there is an arbitrage opportunity. To exploit this, we build a replicating portfolio that buys x_1, x_2, and x_3 units of bonds 1, 2, and 3. We choose our portfolio to replicate the cash-flows of the government bond in each year:

$$\begin{aligned}
\text{year 1} &: 60 = 1080x_1 + 100x_2 + 0x_3 \Rightarrow x_1 = \frac{60 - 100x_2}{1080} = 0.0505 \\
\text{year 2} &: 60 = 0x_1 + 1100x_2 + 0x_3 \Rightarrow x_2 = \frac{60}{1100} = 0.0545 \\
\text{year 3} &: 1060 = 0x_1 + 0x_2 + 1000x_3 \Rightarrow x_3 = \frac{1060}{1000} = 1.06
\end{aligned}$$

Since the traded bond is undervalued, we buy the newly issued government bond and sell our replicating portfolio, that is, we buy one unit of the newly issued bond, sell 0.0505 units of Bond 1, sell 0.0545 units of Bond 2, and sell 1.06 units of Bond 3. The total profit earned today with the strategy is \$975.32 − \$950 = \$25.32.

(c) Denoting the price of Bond 2 at time t by $P_t^{(2)}$, and the future one-year spot rate by ${}_1r_2$, the holding period return next year is:

$$HPR = \frac{100 + P_1^{(2)}}{P_0^{(2)}} - 1 = \frac{100 + \frac{1100}{1+{}_1r_2}}{P_0^{(2)}} - 1$$

Since

$$P_0^{(2)} = \frac{100}{1+0.0643} + \frac{1100}{(1+0.0643)^2} = 1065.06,$$

we have to find the spot rate in a year ${}_1r_2$ such that:

$$\frac{100 + \frac{1100}{1+{}_1r_2}}{1065.06} - 1 > 0 \Rightarrow {}_1r_2 < 13.98\%$$

(d) You can synthetically create a four-year zero coupon bond by buying $1/(1+8\%)$ units of Bond 3, and invest the proceeds at the end of year 3 for another year at a forward rate of $f_4 = 8\%$. Such a portfolio would get you a payoff of zero in years 1 to 3, and a payoff of \$1000 in year 4. The price of the four-year zero coupon bond is equal to the value of the replicating portfolio:

$$P_0^{(4)} = \frac{1000}{(1.07)^3(1.08)} = \$755.83.$$

An equivalent way to get the bond price is to use the forward rate formula to obtain the 4-year spot rate r_4:

$$\begin{aligned}
(1+r_4)^4 &= (1+r_3)^3(1+f_4) \\
\Rightarrow\ (1+r_4)^4 &= (1+0.07)^3(1+0.08) \\
\Rightarrow\ r_4 &= \left[(1.07)^3(1.08)\right]^{\frac{1}{4}} - 1 \\
\Rightarrow\ r_4 &= 0.07249,
\end{aligned}$$

and then compute the four-year zero coupon price:

$$P_0^{(4)} = \frac{1000}{(1.07249\%)^4} = \$755.83.$$

6. (a) The quarterly discount rate is $r_q = (1.1325)^{\frac{1}{4}} - 1 = 0.0316$. Since the dividend grows at a constant rate of $g = 0.016$, we have

$$P_0 = \frac{D_1}{r_q - g} = \frac{D_0(1+g)}{r_q - g} = \frac{0.25(1.016)}{0.0316 - 0.016} = \$16.29.$$

(b) Let P_4 be the expected price of GS a year (four quarters) from today, we have

$$P_4 = \frac{D_5}{r_q - g} = \frac{D_0(1+g)^5}{r_q - g} = \frac{0.25(1.016)^5}{0.0316 - 0.016} = \$17.35.$$

(c) The sum of the expected value of dividends for the next four quarters are

$$D_1+D_2+D_3+D_4 = 0.25(1.016)+0.25(1.016)^2+0.25(1.016)^3+0.25(1.016)^4 = \$1.0406.$$

Since the retention ratio is 10.5%, the sum of expected earnings for the next four quarters is

$$E_1 + E_2 + E_3 + E_4 = \frac{D_1 + D_2 + D_3 + D_4}{1 - 0.105} = \frac{1.0406}{0.895} = \$1.1627.$$

Therefore the price-earnings ratio of GS is $16.29/1.1627 = 14$.

(d) Possible reasons for GS to have a lower P/E ratio than the average P/E ratio of its sector: (i) GS has a lower dividend payout ratio than other firms in the sector, (the lower the expected dividend payout ratio, the lower the P/E ratio), (ii) GS is expected to experience lower growth than other firms in the sector (the lower the expected growth rate, the lower the P/E ratio), (iii) the discount rate applied to GS is higher than the discount rate applied to other firms in the sector (the higher the required rate of return, the lower the P/E ratio.)

(e) Companies that do not pay dividends should not necessarily trade at a value of zero. The Discounted Dividend Model has not been contradicted. Companies that are not paying dividends are reinvesting their earnings and provide returns to the shareholder in terms of capital gains. Further, the fact that a company is not paying a dividend presently or has a current medium- to long-term policy of not paying a dividend does not mean that the company will never pay a dividend in the future. There are also other types of distributions to consider that benefit stockholders, mainly company buy-backs.