Mathematical Finance

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Contents

Part I
Theory

1 Fixed Income Securities

1.1 Definitions

$V_{n,m}$ and $V^g_{n,m}$ denote a fixed income security that pays a cash flow, C, which grows at a rate g that lasts from time, $t = n$ to $t = m$ inclusively. We note the following classical definitions:

a) **Perpetuity**: a fixed income asset in the form of $V_{n,\infty}$.

b) **Annuity**: a fixed income asset in the form of $V_{n,m}$ where $m < \infty$.

c) **Growing Perpetuity**: a fixed income asset in the form of $V^g_{n,\infty}$.

d) **Growing Annuity**: a fixed income asset in the form of $V^g_{n,m}$ where $m < \infty$.

e) **Bond**: A type of Annuity together with a face value, sold through an auction mechanism. We use $B_{y,d}$ to denote a bond that pays cash $\frac{C}{d}$ d-times a year, for y years, with a face value of F.

1.2 Pricing

Before we begin, let us recall the following formula:

$$1 + z + z^2 + z^3 + \cdots = \frac{1}{1-z} \quad \text{if } |z| < 1$$

And also recall the following trick:

$$z^j + z^{j+1} + \cdots = z^j(1 + z + z^2 + \cdots) = \frac{z^j}{1-z} \quad \text{if } |z| < 1$$

1.2.1 Perpetuities and Annuities

We begin by looking a how to price an Annuity: a fixed income asset which pays cash $\frac{C}{(1+r)^i}$ at each time period i for an infinite time horizon ($r > 0$). We thus have:

$$V_{0,\infty} = C\left(1 + \frac{1}{1+r} + \left(\frac{1}{1+r}\right)^2 + \cdots\right) = \sum_{i=0}^{\infty} \frac{C}{(1+r)^i} = C\left(\frac{1}{1-\frac{1}{1+r}}\right) = C\left(\frac{1+r}{r}\right)$$

Comparing this with the classical example of $V_1 = C(\frac{1}{1+r} + (\frac{1}{1+r})^2 + \cdots) = \sum_{i=1}^{\infty} \frac{C}{(1+r)^i} \implies$

$$V_{1,\infty} = V_{0,\infty} - C = C\left(\frac{1+r}{r}\right) - C\left(\frac{1+r}{1+r}\right) = \frac{C}{r}$$

For finite time, i.e., we are looking at an Annuity, which has a value of:

$$V_{n,m} = C\left(\left(\frac{1}{1+r}\right)^n + \left(\frac{1}{1+r}\right)^{n+1} + \cdots + \left(\frac{1}{1+r}\right)^m\right) \quad \text{where } n < m$$

If we note that:

$$V_{n,m} = V_{n,\infty} - V_{m+1,\infty}$$

by our trick in 1.2, we can calculate:

$$V_{n,m} = \left(\frac{1}{(1+r)^n}\right)V_{0,\infty} - \left(\frac{1}{(1+r)^{m+1}}\right)V_{0,\infty} = \frac{C}{r}\left(\frac{(1+r)^{m-n+1}-1}{(1+r)^m}\right)$$

Which when $n = 1$, collapses down to the classical equation of:

$$V_{1,m} = \frac{C}{r}\left(1 - \frac{1}{(1+r)^m}\right)$$

1.2.2 Growing Perpetuities and Annuities

Both the Growing Perpetuity and Growing Annuity are completely analogous to their non-growth versions, but the cash payments are now equal to $C(\frac{(1+g)^{i-1}}{(1+r)^i})$ in time period i ($i \geq 1$ and $g < r$). Therefore, for a Growing Perpetuity:

$$V^g_{0,\infty} = C\left(1 + \frac{1}{1+r} + \frac{1+g}{(1+r)^2} + \cdots\right)$$

And hence,

$$V^g_{0,\infty} = \frac{C}{1+g}\left(\sum_{i=0}^{\infty}\left(\frac{1+g}{1+r}\right)^i - g\right) = \frac{C}{1+g}\left(\frac{1}{1-\frac{1+g}{1+r}} - g\right) = C\left(\frac{1+r-g}{r-g}\right)$$

And for the classical example:

$$V^g_{1,\infty} = C\left(\frac{1}{1+r} + \frac{1+g}{(1+r)^2} + \frac{(1+g)^2}{(1+r)^3} + \cdots\right) = V^g_{0,\infty} - C = C\left(\frac{1}{r-g}\right)$$

For finite time, i.e., we have a Growing Annuity:

$$V^g_{n,m} = \left[\left(\frac{1+g}{1+r}\right)^{n-1} - \left(\frac{1+g}{1+r}\right)^m\right]V^g_{1,\infty} = \frac{C}{r-g}\left[\left(\frac{1+g}{1+r}\right)^{n-1} - \left(\frac{1+g}{1+r}\right)^m\right]$$

And hence if $n = 1$, we have the classical equation:

$$V^g_{1,m} = \frac{C}{r-g}\left[1 - \left(\frac{1+g}{1+r}\right)^m\right]$$

1.2.3 Continuous Extensions

We adapt our finite time horizon formulae for fixed income securities to work with continuous compounding by defining: $\hat{r} = \frac{r}{d}$, $\hat{g} = \frac{g}{d}$, $\hat{C} = \frac{C}{d}$, $\hat{m} = dm$, $\hat{n} - 1 = d(n-1)$ and taking the limit as $d \to \infty$. We let $\hat{V}^g_{n,m}$ denote a continuously compounded fixed income security as. We now find the general case for a growing continuous compounded annuity, and note all other cases are corollaries of this (take $g = 0$, or $n = 1$).

$$\hat{V}^g_{n,m} = \lim_{d\to\infty} V^g_{n,m} = \lim_{d\to\infty} \frac{C/d}{(r-g)/d}\left[\left(\frac{1+g/d}{1+r/d}\right)^{d(n-1)} - \left(\frac{1+g/d}{1+r/d}\right)^{md}\right]$$

And since $e^a = \lim_{j\to\infty}(1+\frac{a}{j})^j$, it $\implies$

$$\hat{V}^g_{n,m} = \frac{C}{r-g}\left(e^{(g-r)(n-1)}\left[1-e^{(g-r)(m-n+1)}\right]\right)$$

1.2.4 Summary

Discrete Infinite Time Horizon Pricing at time $n = 0$:

$$V_{0,\infty} = C\left(\frac{1+r}{r}\right) \tag{1}$$

$$V^g_{0,\infty} = C\left(\frac{1+r-g}{r-g}\right) \tag{2}$$

Discrete Infinite Time Horizon Pricing at time $n = 1$:

$$V_{1,\infty} = \frac{C}{r} \tag{3}$$

$$V^g_{1,\infty} = \frac{C}{r-g} \tag{4}$$

Discrete and Continuous Finite Time Horizon Pricing at time $n = 1$:

$$V_{1,m} = \frac{C}{r}\left(1 - \frac{1}{(1+r)^m}\right) \tag{5}$$

$$V^g_{1,m} = \frac{C}{r-g}\left[1 - \left(\frac{1+g}{1+r}\right)^m\right] \tag{6}$$

$$\hat{V}_{1,m} = \frac{C}{r}\left(1 - e^{-rm}\right) \tag{7}$$

$$\hat{V}^g_{1,m} = \frac{C}{r-g}\left(1 - e^{(g-r)(m)}\right) \tag{8}$$

Discrete and Continuous Finite Time Horizon Pricing at arbitrary starting time:

$$V_{n,m} = \frac{C}{r}\left(\frac{(1+r)^{m-n+1} - 1}{(1+r)^m}\right) \tag{9}$$

$$V^g_{n,m} = \frac{C}{r-g}\left[\left(\frac{1+g}{1+r}\right)^{n-1} - \left(\frac{1+g}{1+r}\right)^m\right] \tag{10}$$

$$\hat{V}_{n,m} = \frac{C}{r}\left(e^{-r(n-1)}\left[1 - e^{-r(m-n+1)}\right]\right) \tag{11}$$

$$\hat{V}^g_{n,m} = \frac{C}{r-g}\left(e^{(g-r)(n-1)}\left[1 - e^{(g-r)(m-n+1)}\right]\right) \tag{12}$$

1.3 Bonds

Please reference our definition of a Bond.

1.3.1 Discrete Pricing

We set up the price of a bond, $B_{y,d}$, as follow:

$$B_{y,d} = \frac{F}{(1+\frac{\lambda}{d})^{dy}} + \sum_{i=1}^{dy} \frac{C/d}{(1+\frac{\lambda}{d})^{i}}$$

Where $F :=$ the Face Value, $C :=$ Annual Coupon Payment, $d :=$ the number of times compounding happens per year, $\lambda :=$ the Yield To Maturity.

Since the right hand side of our equation is analogous to an example of a fixed income annuity, we make use of section 1.2's findings to yield:

$$B_{y,d} = \frac{F}{(1+\frac{\lambda}{d})^{dy}} + \frac{C}{\frac{\lambda}{d}d}\left(1 - \frac{1}{(1+\frac{\lambda}{d})^{dy}}\right) = \left(F + \frac{C}{\lambda}\left[\left(1+\frac{\lambda}{d}\right)^{dy} - 1\right]\right)\left(1+\frac{\lambda}{d}\right)^{-dy}$$

1.3.2 Continuous Pricing

We simply take the limit as $d \to \infty$, and by the same derivation of $\hat{V}_{1,m}$, where $y = m$ and $\hat{\lambda} = r$:

$$\hat{B}_y := \lim_{d\to\infty} B_{y,d} = \lim_{d\to\infty}\left(\frac{F}{(1+\frac{\hat{\lambda}}{d})^{dy}} + \frac{C}{\frac{\hat{\lambda}}{d}d}\left[1 - \frac{1}{(1+\frac{\hat{\lambda}}{d})^{dy}}\right]\right) = Fe^{-\hat{\lambda}y} + \frac{C}{\hat{\lambda}}\left(1 - e^{-\hat{\lambda}y}\right)$$

1.3.3 T-Bills

In the context of government issued bonds (T-Bills), the industry defines the spot rate as what we already saw to be λ given the T-Bill is a zero-coupon bond ($C = 0$). If this is the case, then:

$$B_{y,d} = \frac{F}{(1+\frac{\lambda}{d})^{dy}} \iff \lambda = d\left[\left(\frac{F}{B_{y,d}}\right)^{\frac{1}{dy}} - 1\right]$$

And if our T-Bill is compounded continuously:

$$\hat{B}_y = Fe^{-\hat{\lambda}y} \iff \hat{\lambda} = \frac{\log\left(\frac{F}{\hat{B}_y}\right)}{y}$$

We have now just about developed the necessary tools to be able to devise an algorithm for calculating the yield curve (a plot of spot rates). This algorithm is called "bootstrapping". But before we introduce it, we need to define "Forward Rates".

1.3.4 Forward Rates

In the context of zero coupon bonds, given $V_{y,d}$ and $V_{x,d}$ $(x < y)$ one can easily derive an implied spot rate for any zero coupon bond from the future time x to y. This is called the forward rate from x to y, which we will denote $\lambda_{(y-x)}$. Due a no-arbitrage market, we get the equation:

$$B_{y,d} = F\left(1+\frac{\lambda_y}{d}\right)^{-dy} = F\left(1+\frac{\lambda_x}{d}\right)^{-dx}\left(1+\frac{\lambda_{(y-x)}}{d}\right)^{-d(y-x)}$$

And solving for $\lambda_{(y-x)}$:

$$\lambda_{(y-x)} = d\left(\left[\left(1+\frac{\lambda_y}{d}\right)^{-dy}\left(1+\frac{\lambda_x}{d}\right)^{dx}\right]^{\frac{1}{d(x-y)}} - 1\right)$$

If we plug in our implied λ_x and λ_y from 1.3.3, then:

$$\lambda_{(y-x)} = d\left(\left[\left(\left(\frac{F}{B_{y,d}}\right)^{\frac{1}{dy}}\right)^{-dy}\left(\left(\frac{F}{B_{x,d}}\right)^{\frac{1}{dx}}\right)^{dx}\right]^{\frac{1}{d(x-y)}} - 1\right) = d\left[\left(\frac{B_{y,d}}{B_{x,d}}\right)^{\frac{1}{d(x-y)}} - 1\right]$$

And by similar derivation, if our bonds were being continuously compounded:

$$\hat{\lambda}_{(y-x)} = \frac{y\hat{\lambda}_y - x\hat{\lambda}_x}{y-x}$$

And if $x = \alpha y, \alpha \in (0,1)$, then:

$$\hat{\lambda}_{(1-\alpha)y} = \frac{\hat{\lambda}_y - \alpha\hat{\lambda}_{\alpha y}}{1-\alpha}$$

If we plug in our implied $\hat{\lambda}_{\alpha y}$ and $\hat{\lambda}_y$ from section 1.3.3, then:

$$\hat{\lambda}_{(1-\alpha)y} = \frac{\log\left(\frac{\hat{B}_{\alpha y}}{\hat{B}_y}\right)}{y(1-\alpha)}$$

1.3.5 Bootstrapping

"Bootstrapping" is an algorithm for determining the spot rates for a given set of coupon bearing bonds. This algorithm is defined as follows:

Given a bond which pays coupons every d times a year, we would like to solve for $r_1, r_2, \ldots, r_d, \ldots, r_{dy}$ (the first $d \cdot y$ spot rates). We do this by first obtaining r_k, $k < i$, then inputting this to solve with these values for r_i, i.e in our continuous time model:

$$r_i = \frac{-\log\left(\frac{P(0,i) - \sum_{0<k<i} C_i e^{-k \cdot r_k}}{C_i + V_i}\right)}{i}$$

2 Options

2.1 Definitions

An option is a contract which gives the holder the ability to exercise a payoff function $f(\mathbf{a})$ ($\mathbf{a}$ is a vector of assets) within some time period $T \subset \mathbb{R}^+$. We denote a generic option as $O(f,T)$. We now state the following classical definitions:

a) **Callable Option**: an option $O(f,T)$ where $f(\mathbf{a}) = \max(\mathbf{a}_t - K, 0)$ and where $\mathbf{a}_t$ is $\mathbf{a}$'s price at time $t \in T$, and K is a "strike price". For simplicity of notation, we refer to this a callable option as C.

b) **Puttable Option**: an option $O(f,T)$ where $f(\mathbf{a}) = \max(K - \mathbf{a}_t, 0)$ and where $\mathbf{a}_t$ and K is as above. For simplicity of notation, we refer to this a puttable option as P.

c) **American Option**: an option $O(f,T)$ where $T = \{t \mid 0 \leq t \leq \mathcal{T}, \mathcal{T} \neq 0\}$

d) **European Option**: an option $O(f,T)$ where $T = t \in \mathbb{R}^+$

2.2 Arbitrage Pricing

Theorem. 2.1: The Put Call Parity

Let C denote the price of a callable European option at $t = 0$, and P denote the price of a puttable European option at $t = 0$, both with the same strike price K and $T_C = T_P = T$, then:

$$C - P = \mathbf{a}_0 - \frac{K}{(1+r)^T}$$

Proof. We know the payoff for C and P will be $\max(\mathbf{a}_T - K, 0)$ and $\max(K - \mathbf{a}_T, 0)$. If we construct an option $O(f,T) = C - P$, the payoff will be:

$$\begin{aligned} f(\mathbf{a}) &= \max(\mathbf{a}_T - K, 0) - \max(K - \mathbf{a}_T, 0) \\ &= \max(\mathbf{a}_T - K, 0) + \min(\mathbf{a}_T - K, 0) \\ &= \mathbf{a}_T - K \end{aligned}$$

Therefore, if we discount our payoff of $O(f,T)$ back to $t = 0$, we see that $C - P = \mathbf{a}_0 - \frac{K}{(1+r)^T}$. □

Corollary. 2.1: Bounds for Option Pricing* (Working)**

1. $C \geq \mathbf{a}_0 - \frac{K}{(1+r)^T}$ and $C \leq \mathbf{a}_0$

2.3 A Review of Probability Spaces

Definition. 2.1: Probability Space

A Probability Space is a measure space such that the measure over the entire space is equal to 1. More explicitly, a Probability Space is a triple, $(\Omega, X, \mathbb{P})$, consisting of:

1. A non-empty sample space, denoted Ω, (the set of all possible outcomes).
2. The power set of Ω, denoted $X = \mathcal{P}(\Omega)$, called events, such that:
   (a) $\Omega \in X$
   (b) If $\mathbf{x} \in X$, then $(\Omega \backslash \mathbf{x}) = \mathbf{x}^c \in X$
   (c) If $\mathbf{x}_1, \mathbf{x}_2, \dots \in X$, then $\cup_{i=1}^{\infty} \mathbf{x}_i \in X$
3. A probability measure, $\mathbb{P} : X \to [0, 1]$ such that:
   (a) $\mathbb{P}(X \in \Omega) = 1$
   (b) If $\{\mathbf{x}_i\}_{i=1}^{\infty} \subseteq X$ and $\mathbf{x}_i \cap \mathbf{x}_j = \emptyset \ \forall i, j \in \mathbb{N}, i \neq j$, then $\mathbb{P}(X \in \{\mathbf{x}_i\}_{i=1}^{\infty}) = \sum_{i=1}^{\infty} \mathbb{P}(X \in x_i)$

Some examples are:

1. The Binomial Distribution $B(n, p)$:

$$\left(N = \{0, 1, \dots, n\} \ , \ X = \{[k, l] : 0 \leq k \leq l \leq n, \} \ , \ \mathbb{P}\big(X \in [k, l]\big) = \sum_{i=\lceil k \rceil}^{\lfloor l \rfloor} \binom{n}{i} p^i (1-p)^{n-i} \right)$$

2. The Uniform Distribution $U(\alpha, \beta)$:

$$\left([\alpha, \beta] \ , \ X = \{(x, y) : \alpha < x \leq y < \beta\} \ , \ \mathbb{P}\big(X \in (x, y)\big) = \frac{y - x}{\beta - \alpha} \right)$$

3. The Normal Distribution $N(\mu, \sigma)$:

$$\left(\mathbb{R} \ , \ X = \{(x, y) : -\infty < x \leq y < \infty\} \ , \ \mathbb{P}\big(X \in (x, y)\big) = \int_x^y \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{(z-\mu)}{\sigma}\right)^2} dz \right)$$

2.4 Binomial Pricing

2.4.1 2 States, 1 Period, No Interest

We first consider the simplest of options: An option $O(f, T)$, where $T = 1$;

$$f(\mathbf{a}) = \begin{cases} u & \text{where } \mathbb{P}(u) = p \\ d & \text{where } \mathbb{P}(d) = 1 - p \end{cases}$$

And $V = \mathbb{E}(O) = p \cdot u + (1-p) \cdot d$.

As an example, consider $V = 1, u = 2, d = .5$, (V denotes the value of the option at $t = 0$. we can find $\mathbb{P}(u) = p$ from the formula:

$$p = \frac{\mathbb{E}(O) - d}{u - d} = \frac{1 - .5}{2 - .5} = \frac{1}{3}$$

2.4.2 2 States, 1 Period, With Interest

We now extend our above description to include interest. We consider the same set up for $O(f, T)$ as above, except now: $(1 + r)V = \mathbb{E}(O) = p \cdot u + (1 - p) \cdot d$, or in the continuous time setting, $e^r \cdot V = \mathbb{E}(O) = p \cdot u + (1 - p) \cdot d$.

As an example, consider $V = 1, u = 2, d = .5, r = .1$, we can find $\mathbb{P}(u) = p$ from the formula:

$$p = \frac{V(1 + r) - d}{u - d} = \frac{1(1 + .1) - .5}{2 - .5} = \frac{2}{5}$$

2.4.3 4 States, 2 Periods, No Interest (Working Example)

We now consider an option $O(f, T)$, where $T = 2$;

$$f(\mathbf{a}) = \begin{cases} uu & \text{where } \mathbb{P}(uu) = \mathbb{P}(u) \cdot \mathbb{P}(u) = p^2 \\ ud & \text{where } \mathbb{P}(ud) = \mathbb{P}(u) \cdot \mathbb{P}(d) = p(1-p) \\ du & \text{where } \mathbb{P}(du) = \mathbb{P}(d) \cdot \mathbb{P}(u) = (1-p)p \\ dd & \text{where } \mathbb{P}(dd) = \mathbb{P}(d) \cdot \mathbb{P}(d) = (1-p)^2 \end{cases}$$

And $\mathbb{E}(O) = p \cdot u + (1-p) \cdot d$.

As an example, consider $V = 1, uu = 4, ud = du = 1, d = .25$. We can find $\mathbb{P}(u) = p$ from the formula:

$$p = \frac{2(dd - ud) + 2\sqrt{(ud - dd)^2 - (uu - 2ud + dd)(dd - V)}}{uu - 2ud + dd} = \frac{-2}{3} + \frac{4}{3} = \frac{2}{3} \text{ or } \frac{1}{3}?$$

2.5 General Calls and Puts Without Interest

Assume our option $O(f, T)$ may either go up or down by $100 \cdot \alpha\%$ ($\alpha \in (0, 1)$), and is originally priced at $S_0 = 1$. Then a Put without interest will be valued by:

$$V_{Put} = \max_{0 \le t \le T} \left(\sum_{j=0}^{\lfloor t/2 \rfloor} \binom{t}{j} \left(\frac{1 - (1-\alpha)^{t-j}(\alpha)^j}{2^t} \right) \right)$$

And similarly for a Call:

$$V_{Call} = \max_{0 \le t \le T} \left(\sum_{j=\lceil t/2 \rceil}^{t} \binom{t}{j} \left(\frac{(1-\alpha)^{t-j}(\alpha)^j - 1}{2^t} \right) \right)$$

2.6 General Calls and Puts With Interest

If we add (continuous) interest to our above equations, we get:

$$V_{Put} = \max_{0 \le t \le T} \left(\sum_{j=0}^{\lfloor e^{-rt} t/2 \rfloor} \binom{t}{j} \left(\frac{1 - e^{-rt}(1-\alpha)^{t-j}(\alpha)^j}{2^t} \right) \right)$$

And similarly for a Call:

$$V_{Call} = \max_{0 \le t \le T} \left(\sum_{j=\lceil e^{-rt} t/2 \rceil}^{t} \binom{t}{j} \left(\frac{e^{-rt}(1-\alpha)^{t-j}(\alpha)^j - 1}{2^t} \right) \right)$$

2.7 Brownian Motion

Definition. 2.2: Brownian Motion

Brownian Motion is a process $\{B_t\}_{t \ge 0}$ s.t such that $B_t : \mathbb{R}^+ \to \mathbb{R}$, $B_t \in C(\mathbb{R}^+)$, and:

1. $B_t \sim N(0, t)$
2. If $0 \le t_1 \le s_1 \le t_2 \le \cdots \le t_k \le s_k$, then $B_{s_i} - B_{t_i} \sim N(0, s_i - t_i)$ and $\{B_{s_i} - B_{t_i}\}_{i=1}^k$ are all independent.
3. $\text{Cov}(B_s, B_t) = \min(s, t)$

Example. 2.1: Variance of The Product of a Brownian Motion

Let $\{B_t\}_{t \ge 0}$ and $\{B_s\}_{s \ge 0}$ be Brownian motion. Find an explicit formula for $\mathbb{V}\text{ar}(B_s B_t)$

Answer: Assume that $0 \le t \le s$, then:

$$\begin{aligned}
\mathbb{V}\text{ar}(B_s B_t) &= \mathbb{E}\left[(B_s B_t)^2\right] - \left[\mathbb{E}(B_s B_t)\right]^2 \\
&= \mathbb{E}\big((B_t(B_s - B_t + B_t))^2\big) - \big(\mathbb{E}(B_t(B_s - B_t + B_t))\big)^2 \\
&= \mathbb{E}\big((B_t)^2(B_s - B_t)^2 + 2B_t(B_s - B_t) + (B_t)^4\big) - \big(\mathbb{E}(B_t(B_s - B_t)) + \mathbb{E}((B_t)^2)\big)^2 \\
&= \mathbb{E}\big((B_t)^2\big)\mathbb{E}\big((B_s - B_t)^2\big) + 2\mathbb{E}\big(B_t\big)\mathbb{E}\big((B_s - B_t)\big) + \mathbb{E}\big((B_t)^4\big) \\
&\qquad\qquad - \big(\mathbb{E}(B_t)\mathbb{E}(B_s - B_t) + \mathbb{E}((B_t)^2)\big)^2 \\
&= t(s - t) + 2(0)(0) + t^2 - \big((0)(0) + (t)\big)^2 \\
&= t \cdot (t + s)
\end{aligned}$$

And this implies (if we only assume $s, t \ge 0$):

$$\mathbb{V}\text{ar}(B_s B_t) = \min(s, t) \cdot (s + t)$$

Corollary. 2.2: Diffusion Equations

Suppose $X_t = x_0 + \mu t + \sigma B_t$ where $\{B_t\}_{t \geq 0}$ is Brownian Motion. Then $X_t \sim N(x_0 + \mu t, \sigma^2 t)$, and $\mathbb{C}\text{ov}(X_t, X_s) = \sigma^2 \min(s,t)$. We write this as $dX_t = \mu dt + \sigma dB_t$, where $\mu :=$ "drift", $\sigma :=$ "volatility", and $\sigma \geq 0$.

Proof.

$$\mathbb{E}(X_t) = \mathbb{E}(x_0 + \mu t + \sigma B_t) = \mathbb{E}(x_0) + \mu\mathbb{E}(t) + \sigma\mathbb{E}(B_t) = x_0 + \mu t$$

$$\mathbb{V}\text{ar}(X_t) = \mathbb{V}\text{ar}(x_0 + \mu t + \sigma B_t) = \sigma^2 \mathbb{V}\text{ar}(B_t) = \sigma^2 t$$

$$\mathbb{C}\text{ov}(X_s, X_t) = \mathbb{E}((X_t - \mu t - x_0)(X_s - \mu t - x_0)) = \mathbb{E}(\sigma^2 B_t B_s) = \sigma^2 \mathbb{E}(B_t B_s) = \sigma^2 \min(s,t)$$

□

Example. 2.2:

Let $\{B_t\}_{t \geq 0}$ be Brownian motion. Compute $\mathbb{V}\text{ar}(B_5 B_8)$. [Necessary Lemma: If $Z \sim \text{Normal}(0,1)$, then $\mathbb{E}(Z) = \mathbb{E}(Z^3) = 0$, $\mathbb{E}(Z^2) = 1$, and $\mathbb{E}(Z^4) = 3$.]

Answer: We first solve the general case below for B_s and B_t (and assume $0 \leq t \leq s$):

$$\begin{aligned}
\mathbb{V}\text{ar}(B_s B_t) &= \mathbb{E}\big[(B_s B_t)^2\big] - \big[\mathbb{E}(B_s B_t)\big]^2 \\
&= \mathbb{E}\big((B_t(B_s - B_t + B_t))^2\big) - \big(\mathbb{E}(B_t(B_s - B_t + B_t))\big)^2 \\
&= \mathbb{E}\big((B_t)^2(B_s - B_t)^2 + 2(B_t)^3(B_s - B_t) + (B_t)^4\big) - \big(\mathbb{E}(B_t(B_s - B_t)) + \mathbb{E}((B_t)^2)\big)^2 \\
&= \mathbb{E}\big((B_t)^2\big)\mathbb{E}\big((B_s - B_t)^2\big) + 2\mathbb{E}\big((B_t)^3\big)\mathbb{E}\big((B_s - B_t)\big) + \mathbb{E}\big((B_t)^4\big) \\
&\qquad - \big(\mathbb{E}(B_t)\mathbb{E}(B_s - B_t) + \mathbb{E}((B_t)^2)\big)^2 \\
&= t(s - t) + 2(0)(0) + 3t^2 - \big((0)(0) + (t)\big)^2 \\
&= t \cdot (t + s)
\end{aligned}$$

And this implies (if we only assume $s, t \geq 0$):

$$\mathbb{V}\text{ar}(B_s B_t) = \min(s,t) \cdot (s + t)$$

Therefore, we have here that:

$$\mathbb{V}\text{ar}(B_5 B_8) = \min(5,8) \cdot (8 + 5) = 65$$

2.8 Martingales

Definition. 2.3: Martingale

A sequence $\{X_n\}_{n=0}^{\infty}$ of random variables is a martingale if $\mathbb{E}|X_n| < \infty \ \forall n \in \mathbb{N} \cup \{0\}$, and also $\mathbb{E}(X_{n+1}|X_0, \ldots, X_n) = X_n$. I.e., X_n's average is constant $\forall n$. Furthermore, if a Markov Chain has the property that: $\sum_{j \in S} j p_{ij} = i \ \forall i \in S \implies$ it is a martingale.

Example. 2.3: (Unmotivated*)

Prove that the Markov chain defined by the graph, $(V, w(V))$, is martingale, where:

$$V = \mathbb{Z}, \text{ and } w(i,j) = \begin{cases} 1 & \text{if } 0 < |i-j| \leq n \\ 0 & \text{otherwise} \end{cases}$$

*skip to next example if unfamiliar with Markov Chains or Basic Graph Theory

Proof. We check that $\sum_{j \in S} j \cdot p_{ij} = i \; \forall i \in S$ (also note that $d(i) = \sum_{i \in \mathbb{Z}} w(i,j) = 2n$) :

$$\begin{aligned}
\sum_{j \in S} j \cdot p_{ij} &= \sum_{j \in \mathbb{Z}} j \cdot \left(\mathbb{1}_{(0<|i-j|\leq n)} \frac{1}{2n} \right) \\
&= \sum_{j \in \{i \pm k, 0<k\leq n\}} j \cdot \left(\frac{1}{2n} \right) \\
&= \sum_{k=1}^{n} 2(i \pm k) \left(\frac{1}{2n} \right) \\
&= \frac{1}{n} \sum_{k=1}^{n} i + \sum_{k=1}^{n} \frac{k}{n} - \sum_{k=1}^{n} \frac{k}{n} \\
&= i
\end{aligned}$$

□

Example. 2.4:

Let $\{B_t\}_{t\geq 0}$ be Brownian motion. Let $\theta \in \mathbb{R}$, and let $Z_t = \exp(\theta B_t - \frac{1}{2}\theta^2 t)$. Prove that $\{Z_t\}_{t\geq 0}$ is a martingale. [Necessary Lemma: If $W \sim \text{Normal}(\mu, \sigma^2)$ and $a \in \mathbb{R}$, then $\mathbb{E}[e^{aW}] = e^{\mu a + \sigma^2 a^2/2} < \infty$.]

Proof. Firstly, since we may use the fact that if $W \sim \text{Normal}(\mu, \sigma^2)$, then $\mathbb{E}[e^{aW}] = e^{\mu a + \sigma^2 a^2/2} \implies$ if $W \sim N(\mu t, \sigma^2 t)$ then:

$$\mathbb{E}[e^{aW}] = e^{\mu a t + \frac{1}{2}\sigma^2 a^2 t} \tag{13}$$

Next, we note that $\{Z_t\}_{t\geq 0}$ is non-anticipating since $\{B_t\}_{t\geq 0}$ is non-anticipating.

Furthermore, we can see that $\mathbb{E}|Z_n| = 1 < \infty$ since:

$$\begin{aligned}
\mathbb{E}\big[Z_t\big] &= \mathbb{E}\big[e^{\theta B_t - \frac{1}{2}\theta^2 t}\big] \\
&= \mathbb{E}\big[e^{\theta B_t} e^{-\frac{1}{2}\theta^2 t}\big] \\
&= e^{-\frac{1}{2}\theta^2 t}\mathbb{E}\big[e^{\theta B_t}\big] \\
&= e^{-\frac{1}{2}\theta^2 t}\big[e^{0+\frac{1}{2}(1)^2(\theta)^2 t}\big] && \text{since } B_t \sim \text{Normal}(0,t) \text{ and by Eq.(1) above} \\
&= 1
\end{aligned}$$

Let us define $\Delta B_{t+h} := B_{t+h} - B_t$, then:

$$
\begin{aligned}
\mathbb{E}[Z_{t+h}|Z_t] &= \mathbb{E}\Big[Z_{t+h}\Big(\frac{Z_t}{Z_t}\Big)\Big|Z_t\Big] \\
&= \mathbb{E}\Big[Z_t\Big(\frac{Z_{t+h}}{Z_t}\Big)\Big|Z_t\Big] \\
&= \mathbb{E}\Big[e^{\left(\theta B_t - \frac{1}{2}\theta^2 t\right)}\Big(e^{\left[\left(\theta B_{t+h} - \frac{1}{2}\theta^2 (t+h)\right) - \left(\theta B_t - \frac{1}{2}\theta^2 t\right)\right]}\Big)\Big|Z_t\Big] \\
&= \mathbb{E}\Big[e^{\left(\theta B_t - \frac{1}{2}\theta^2 t\right)}\Big(e^{\left(\theta(B_{t+h} - B_t) - \frac{1}{2}\theta^2 h\right)}\Big)\Big|Z_t\Big] \\
&= \mathbb{E}\Big[e^{\left(\theta B_t - \frac{1}{2}\theta^2 t\right)}\Big|Z_t\Big]\cdot\mathbb{E}\Big[e^{\left(\theta(B_{t+h} - B_t) - \frac{1}{2}\theta^2 h\right)}\Big] && \text{since } B_t \text{ indep. of } B_{t+h} - B_t \\
&= e^{\left(\theta B_t - \frac{1}{2}\theta^2 t\right)}\cdot\mathbb{E}\Big[e^{\left(\theta(B_{t+h} - B_t) - \frac{1}{2}\theta^2 h\right)}\Big] && \text{since } B_t \text{ is non-anticipating} \\
&= Z_t \cdot e^{-\frac{1}{2}\theta^2 h}\mathbb{E}\Big[e^{\theta(B_{t+h} - B_t)}\Big] && \text{since } e^{-\frac{1}{2}\theta^2 h} \text{ is constant} \\
&= Z_t \cdot e^{-\frac{1}{2}\theta^2 h} e^{0 + \frac{1}{2}(1)^2\theta^2 h} && \text{since } B_{t+h} - B_t \sim \mathrm{N}(0,h) \\
&= Z_t && \text{and Eq. (13)}
\end{aligned}
$$

And hence we have now shown all three necessary properties for a random variable to be martingale.

□

2.9 Deriving the Black-Scholes Formula

Theorem. 2.2: Black-Scholes Formula

Let $\{B_t\}_{t\geq 0}$ be Brownian motion. Additionally, let $X_t := x_0 \exp(\mu t + \sigma B_t)$ be our stock price model (where $\sigma > 0$), and let $D_t = e^{-rt}X_t$ be the discounted stock price. Then we have the following two results:

(a) If $\mu = r - \frac{\sigma^2}{2}$, then $\{D_t\}$ is a martingale. [Necessary Lemma: Example 2.4]

(b) If $\mu = r - \frac{\sigma^2}{2}$, then:

$$
\begin{aligned}
\mathbb{E}\big[e^{-rS}\max(0, X_S - K)\big] = X_0\Phi&\left(\frac{(r + \frac{\sigma^2}{2})S - \log(K/X_0)}{\sigma\sqrt{S}}\right) \\
&- e^{-rS}K\Phi\left(\frac{(r - \frac{\sigma^2}{2})S - \log(K/X_0)}{\sigma\sqrt{S}}\right)
\end{aligned}
$$

where $\Phi(u) = \int_{-\infty}^{u}\frac{1}{\sqrt{2\pi}}e^{-v^2/2}dv$ is the cdf of a standard normal distribution.

This is the famous "Black-Scholes Formula".

Proof.

(a) We first simplify D_t if $\mu = r - \frac{\sigma^2}{2}$:

$$D_t := e^{-rt}x_0 e^{\mu t + \sigma B_t} = x_0 e^{-rt + (r - \frac{\sigma^2}{2})t + \sigma B_t} = x_0 e^{\sigma B_t - \frac{1}{2}\sigma^2 t}$$

And since $D_t = x_0 Z_t$ where $Z_t := \exp(\theta B_t - \frac{1}{2}\theta^2 t), \theta \in \mathbb{R}$. We thus must leverage the previous question to prove that $D_t = x_0 Z_t$ is martingale. D_t being non-anticipating and having $\mathbb{E}|D_t| < \infty$ follow trivially since (respectively) $\{B_t\}_{t\geq 0}$ is non-anticipating and $\mathbb{E}[D_t] = x_0\mathbb{E}[Z_t] = x_0$. The fact that $\mathbb{E}[D_{t+h}|D_t] = D_t$ follows also pretty trivially since when referencing the previous question, we can see: $\mathbb{E}[D_{t+h}|D_t] = x_0[Z_{t+h}|Z_t] = x_0 Z_t = D_t$. Thus, we have now shown all three necessary properties for this random variable to be martingale.

(b) **Answer:** Our strategy will be as follows: We first write the expectation as an integral with respect to the density function for B_S. Then, break up the integral into the part where $X_S - K \geq 0$ and the part where $X_S - K < 0$.

To begin, we recall that $B_t \sim \text{Normal}(0,t)$, hence if we let $B_t = \sqrt{t}Y$, then we have:

$$X_t = x_0 \exp\big(\mu t + \sigma B_t\big) = x_0 \exp\big(\mu t + \sigma\sqrt{t}Y\big), \quad \text{where } Y \sim \text{Normal}(0,1)$$

We can now do this (quite lengthily) computation directly:

$$\begin{aligned}\mathbb{E}\big[e^{-rt}\max(0, X_S - K)\big] &= e^{-rt}\mathbb{E}\big[\max(0, X_S - K)\big] \\ &= e^{-rt}\int_{-\infty}^{\infty} \max\big(0, x_0 e^{\mu t + \sigma\sqrt{t}y} - K\big)\frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy\end{aligned}$$

Now, we note that:

$$\max\big(0, x_0 e^{\mu t+\sigma\sqrt{t}y} - K\big) = \begin{cases} x_0 e^{\mu t+\sigma\sqrt{t}y} - K & \text{if } y > \frac{\log(K/x_0)-\mu t}{\sigma\sqrt{t}} \\ 0 & \text{if } y \leq \frac{\log(K/x_0)-\mu t}{\sigma\sqrt{t}} \end{cases}$$

And hence we may proceed in the following fashion (also let $z := \frac{\log(K/x_0)-\mu t}{\sigma\sqrt{t}}$):

$$\begin{aligned}\mathbb{E}\big[e^{-rt}\max(0, X_S - K)\big] &= e^{-rt}\int_{-\infty}^{\infty} \max\big(0, x_0 e^{\mu t+\sigma\sqrt{t}y} - K\big)\frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy \\ &= e^{-rt}\int_{-\infty}^{z}(0)\frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy \\ &\quad + e^{-rt}\int_{z}^{\infty}\Big(x_0 e^{\mu t+\sigma\sqrt{t}y} - K\Big)\frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy \\ &= \left(e^{-rt}x_0\int_z^{\infty}\Big(e^{\mu t+\sigma\sqrt{t}y}\Big)\frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy\right) - \left(e^{-rt}K\int_z^{\infty}\frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy\right) \\ &= \left(e^{-rt}x_0\int_z^{\infty}\frac{1}{\sqrt{2\pi}}e^{\mu t+\sigma\sqrt{t}y-y^2/2}dy\right) - \Big(\big(e^{-rt}K\big(1-\Phi(z)\big)\Big)\end{aligned}$$

Next, we need to perform a little bit of algebra on the argument of $e^{\mu t+\sigma\sqrt{t}y-y^2/2}$ as follows:

$$\begin{aligned}\mu t + \sigma\sqrt{t}y - \frac{y^2}{2} &= \mu t + \sigma\sqrt{t}y - \frac{y^2}{2} - \frac{\sigma^2 t}{2} + \frac{\sigma^2 t}{2} \\ &= \Big(\mu + \frac{1}{2}\sigma^2\Big)t - \Big(\frac{(y-\sigma\sqrt{t})^2}{2}\Big)\end{aligned}$$

And hence we may proceed with our calculations:

$$\begin{aligned}
\mathbb{E}\big[e^{-rt}\max(0, X_S - K)\big] &= \left(e^{-rt}x_0 \int_z^\infty \frac{1}{\sqrt{2\pi}} e^{\mu t + \sigma\sqrt{t}y - y^2/2} dy\right) - \Big((e^{-rt}K\big(1-\Phi(z)\big)\Big) \\
&= \left(e^{-rt}x_0 \int_z^\infty \frac{1}{\sqrt{2\pi}} e^{\left(\mu + \frac{1}{2}\sigma^2\right)t - \left(\frac{(y-\sigma\sqrt{t})^2}{2}\right)} dy\right) - \Big((e^{-rt}K\big(1-\Phi(z)\big)\Big) \\
&= \left(e^{-rt + \left(\mu + \frac{1}{2}\sigma^2\right)t} x_0 \int_z^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-\sigma\sqrt{t})^2)} dy\right) - \Big((e^{-rt}K\big(1-\Phi(z)\big)\Big)
\end{aligned}$$

Now we can substitute $\mu = r - \frac{1}{2}\sigma^2$:

$$\begin{aligned}
\mathbb{E}\big[e^{-rt}\max(0, X_S - K)\big] &= \left(e^{-rt + rt(-\frac{1}{2}\sigma^2 + \frac{1}{2}\sigma^2)t} x_0 \int_z^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-\sigma\sqrt{t})^2)} dy\right) - \Big((e^{-rt}K\big(1-\Phi(z)\big)\Big) \\
&= \left(x_0 \int_z^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-\sigma\sqrt{t})^2)} dy\right) - \Big((e^{-rt}K\big(1-\Phi(z)\big)\Big)
\end{aligned}$$

We can make the change of variables $x = y - \sigma\sqrt{t}$ ($\implies dx = dy$), and hence:

$$\begin{aligned}
\mathbb{E}\big[e^{-rt}\max(0, X_S - K)\big] &= \left(x_0 \int_{z-\sigma\sqrt{t}}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx\right) - \Big((e^{-rt}K\big(1-\Phi(z)\big)\Big) \\
&= \Big(x_0\big(1 - \Phi(z - \sigma\sqrt{t})\big)\Big) - \Big((e^{-rt}K\big(1-\Phi(z)\big)\Big) \\
&= x_0\Phi(\sigma\sqrt{t} - z) - e^{-rt}K\Phi(-z) \\
&= x_0\Phi\left(\sigma\sqrt{t} - \frac{\log(K/x_0) - \mu t}{\sigma\sqrt{t}}\right) - e^{-rt}K\Phi\left(-\frac{\log(K/x_0) - \mu t}{\sigma\sqrt{t}}\right) \\
&= x_0\Phi\left(\frac{\sigma^2 t + \mu t - \log(K/x_0)}{\sigma\sqrt{t}}\right) - e^{-rt}K\Phi\left(\frac{\mu t - \log(K/x_0)}{\sigma\sqrt{t}}\right)
\end{aligned}$$

And substituting in $\mu = r - \frac{1}{2}\sigma^2$ again, we can see:

$$\begin{aligned}
\mathbb{E}\big[e^{-rt}\max(0, X_S - K)\big] &= x_0\Phi\left(\frac{\sigma^2 t + \big(r - \frac{1}{2}\sigma^2\big)t - \log(K/x_0)}{\sigma\sqrt{t}}\right) - e^{-rt}K\Phi\left(\frac{\big(r - \frac{1}{2}\sigma^2\big)t - \log(K/x_0)}{\sigma\sqrt{t}}\right) \\
&= x_0\Phi\left(\frac{\big(r + \frac{1}{2}\sigma^2\big)t - \log(K/x_0)}{\sigma\sqrt{t}}\right) - e^{-rt}K\Phi\left(\frac{\big(r - \frac{1}{2}\sigma^2\big)t - \log(K/x_0)}{\sigma\sqrt{t}}\right)
\end{aligned}$$

□

3 Credit Risk

Forthcoming...

Part II

Classical Examples

4 Examples: Previous APM466 Midterms

4.1 2010

Example. 4.1: 2010 Midterm

1. A stock is valued at \$75 today. An option will pay \$1 the first time the stock reaches \$100 in value, which it is assumed will happen with probability 1 at some point in the future. Find the price of the option, and the replicating portfolio.

2. A stock is valued at \$1 today. In a year, its price S_1 can be worth either \$2 or \$0.50. A convertible bond will pay $\max(1, S_1)$, a year from now. Assuming 0 interest rates, calculate the current price of the convertible bond.

3. With interest rates equal to 0, two different stocks S_1 and S_2, both valued at \$1 today, can be worth \$2 or \$0.50 at some point in the future. If the option that pays \$1 when both $S_1 = S_2 = \$2$ is traded in the market and is worth \$0.125, calculate the price and replicating portfolio of the option that pays \$1 when $S_1 = \$2$ but $S_2 = \$0.5$. (You can leave the answer expressed in matricial form if you prefer).

Answers:

1. We note that $V_{stock} := V = 75$, and since $\mathbb{P}(S_T = 100) = 1 \implies e^{rT}75 = 100 \iff rT = \log\left(\frac{100}{75}\right)$. This implies that: $V_{bond} = e^{-rT} \cdot \mathbb{P}(S_T = 100) \cdot 1 = \frac{75}{100}$

2. By computing $\mathbb{P}(S_1 = \$2)$, we can easily price the bond. We recall the formula $\mathbb{P}(u) = \frac{V-d}{u-d}$ from section 2.4.1, which $\implies p = \frac{1-.5}{2-.5} = \frac{1}{3}$. Therefore, $V_{bond} = \mathbb{E}(bond) = 2p + 1(1-p) = 2\frac{1}{3} + \frac{2}{3} = \frac{4}{3}$.

3. Let $\theta(x)$ denote the payoff of assets $1, \ldots, n$ conditional on the state x being realized, i.e; $D(\omega_i) = \big(\mu(a_1|\omega_i), \ldots, \mu(a_n|\omega_i)\big)$ where $\mu : \mathbf{a} \times X \to \mathbb{R}$. Also, let Θ denote an arbitrary weighting of the assets $1, \ldots, n$. We thus compute:

$$D(\omega_i) = \big(\mu(S_1|\omega_i), \mu(S_2|\omega_i), \mu(B|\omega_i), \mu(O_1|\omega_i)\big)$$

$$\begin{aligned} \implies D(uu) &= (2, 2, 1, 1) \\ D(ud) &= (2, 1/2, 1, 0) \\ D(du) &= (1/2, 2, 1, 0) \\ D(dd) &= (1/2, 1/2, 1, 0) \end{aligned}$$

Therefore, if we solve for Θ in the following equation:

$$\begin{pmatrix} 2 & 2 & 1 & 1 \\ 2 & 1/2 & 1 & 0 \\ 1/2 & 2 & 1 & 0 \\ 1/2 & 1/2 & 1 & 0 \end{pmatrix} \Theta = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

We can conclude that if q denotes a vector of length n of the prices of each asset, then the price of O_2 will be: $\Theta \cdot q$.

To solve the above system of linear equations, we can either solve for the matrix (let us call it D) inverse, D^{-1}, which will give us a general method for determining $\Theta(O_i) \forall i$, or we can transform the matrix $[D|b]$ to it's reduced row echelon form to solve just for $\Theta(O_2)$. If we solve for D^{-1}, we see that:

$$\Theta = D^{-1} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \left(\frac{1}{3}\right) \begin{pmatrix} 0 & 2 & 0 & -2 \\ 0 & 0 & 2 & -2 \\ 0 & -1 & -1 & 5 \\ 3 & -3 & -3 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \left(\frac{1}{3}\right) \begin{pmatrix} 2 \\ 0 \\ -1 \\ -3 \end{pmatrix}$$

Which implies that price of O_2 is $\Theta \cdot (1, 1, 1, 0.125) = \frac{5}{24}$.

If we had solved the problem by performing row reductions to the matrix $[D|b]$, then we see again that:

$$D\Theta = b \iff \left(\begin{array}{cccc|c} 2 & 2 & 1 & 1 & 0 \\ 2 & 1/2 & 1 & 0 & 1 \\ 1/2 & 2 & 1 & 0 & 0 \\ 1/2 & 1/2 & 1 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1/3 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right)$$

Which implies the same value for Θ as was in the D^{-1} case, and hence we also see that the price of O_2 is $\frac{5}{24}$.

4.2 2012

Example. 4.2: 2012 Midterm

1. A market has two stocks, IBB and IMM; IBB is worth \$1.25 today, and after one year it can be worth either \$2 or \$0.5. IMM can also be worth either \$2 or \$0.5 in a year, but we do not know its price today.

   We do know, however, that the option that pays \$1 when both IBB and IMM go up in price (and 0 otherwise) is worth \$0.3, and the option that is pays \$1 when both IBB and IMM go down (and 0 otherwise) is worth \$0.2. Money can be borrowed at 0% per year using a bond.

   Find:

   (a) The price of IMM today.

   (b) The risk neutral joint probabilities for the stock price outcomes.

   (c) The correlation between the two stock prices using the risk neutral measure.

2. With interest rates equal to 0% per quarter, a stock S valued at \$1 today can go up or down in price by 10% each quarter.

   (a) Calculate the arbitrage-free price of an American put option with strike \$1 maturing 9 months from now.

   (b) Determine the values of S and t that will trigger such option to be exercised by the holder before maturity.

   Numerical answers can be rounded to the nearest cent. Algebraic expressions are valid answers if given in explicit form (i.e. numerical calculations are optional, but recommended).

Answers:

1. Let $\theta(x)$ denote the payoff of assets $1, \ldots, n$ conditional on the state x being realized, i.e; $\theta(x) = \big(\mu(a_1|x), \ldots, \mu(a_n|x)\big)$ where $\mu : \mathbf{a} \times X \to \mathbb{R}$. Also, let Θ denote an arbitrary weighting of the assets $1, \ldots, n$. We thus compute:

$$\theta(x) = \big(\mu(S_1|x), \mu(B|x), \mu(O_1|x), \mu(O_2|x)\big)$$

$$\begin{aligned}
\implies \theta(du) &= (1/2, 1, 0, 0) \\
\theta(ud) &= (2, 1, 0, 0) \\
\theta(uu) &= (2, 1, 1, 0) \\
\theta(dd) &= (1/2, 1, 0, 1)
\end{aligned}$$

   Therefore, if we solve for Θ in the following equation:

$$\begin{pmatrix} 1/2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 1/2 & 1 & 0 & 1 \end{pmatrix} \Theta = \begin{pmatrix} 2 \\ 1/2 \\ 2 \\ 1/2 \end{pmatrix}$$

We can conclude that if q denotes a vector of length n of the prices of each asset, then the price of O_2 will be: $\Theta \cdot q$.

To solve the above system of linear equations, we can either solve for the matrix (let us call it D) inverse, D^{-1}, which will give us a general method for determining $\Theta(O_i) \forall i$, or we can transform the matrix $[D|b]$ to it's reduced row echelon form to solve just for $\Theta(O_2)$. If we solve for D^{-1}, we see that:

$$\Theta = D^{-1} \begin{pmatrix} 2 \\ 1/2 \\ 2 \\ 1/2 \end{pmatrix} = \left(\frac{1}{3}\right) \begin{pmatrix} -2 & 2 & 0 & 0 \\ 4 & -1 & 0 & 0 \\ 0 & -3 & 3 & 0 \\ -3 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1/2 \\ 2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} -1 \\ 5/2 \\ 3/2 \\ -3/2 \end{pmatrix}$$

Which implies that price of O_2 is $\Theta \cdot (1.25, 1, 0.3, 0.2) = 1.4$.

If we had solved the problem by performing row reductions to the matrix $[D|b]$, then we see again that:

$$D\Theta = b \iff \left(\begin{array}{cccc|c} 1/2 & 1 & 0 & 0 & 2 \\ 2 & 1 & 0 & 0 & 1/2 \\ 2 & 1 & 1 & 0 & 2 \\ 1/2 & 1 & 0 & 1 & 1/2 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 5/2 \\ 0 & 0 & 1 & 0 & 3/2 \\ 0 & 0 & 0 & 1 & -3/2 \end{array}\right)$$

Which implies the same value for Θ as was in the D^{-1} case, and hence we also see that the price of O_2 is 1.4.

2. We first note that: $\mathbb{P}(S_1 = x, S_2 = y, S_3 = z | S_0 = 1) = \left(\frac{1}{2}\right)^3$ since $p = \frac{1}{2}$ at any given node (quarter), and independence. Next, we compute:

$$\begin{aligned} p_{ddd} &= (.9)^3 = 0.729 \qquad & p_{ddu} = p_{dud} = p_{udd} = (.9)^2(1.1) = 0.891 \\ p_{uuu} &= (1.1)^3 = 1.331 \qquad & p_{uud} = p_{udu} = p_{duu} = (1.1)^2(.9) = 1.089 \end{aligned}$$

Now, we recall from our definition of a puttable option, that $f(\mathbf{a}_T) = \max(K - \mathbf{a}_T, 0)$, and hence we arrive at the formula of

$$V(t=3) = \frac{1}{8}(1 - 0.729) + \frac{3}{8}(1 - 0.891) = 0.07475 = \frac{299}{4000}$$

However, before we finish, we must check that our Put is not more valuable at $t < T = 3$. For $t = 2$, we get that:

$$p_{dd} = (.9)^2 = 0.81 \qquad p_{du} = p_{ud} = (.9)(1.1) = 0.99 \qquad p_{uu} = (1.1)^2 = 1.21$$

And hence the valuation here would be:

$$V(t=2) = \frac{1}{4}(1-0.81) + \frac{1}{2}(1-0.99) = 0.0525 = \frac{210}{4000} < \frac{299}{4000}$$

And similarly for $V(t=1)$:

$$V(t=1) = \frac{1}{2}(1-0.9) = 0.05 = \frac{200}{4000} < \frac{299}{4000}$$

By induction, one can see that $\forall t \in \mathbb{N}$, $V(t < T) < V(t = T)$ (i.e., $V(i) = \frac{1}{20} < \frac{21}{400} < \frac{299}{4000} < \frac{12557}{160000} < \cdots$). Also by induction, one can find the formula for any $n \in \mathbb{N}$:

$$V(t=n) = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{j} \left(\frac{\left(1-(0.9)^{n-j}(1.1)^{j}\right)}{2^n} \right)$$

Part III

Applications

5 Calculating the Yield Curve

Overview

We follow Canadian Government Bonds for a two-week period (January 9-20, 2017), and calculate the yield curve for each day. We compute the yield curves for each day using 10 unique bonds in the context of three different models. Next, we calculate the forward curves and implied forward rates. Finally, we take three time series of interest and compute their respective covariance matrices, and the eigenvalues and eigenvectors associated with said matrices.

Preface:

Our methodology for this assignment goes as follows:

- We begin by computing the main questions under the assumptions of continuous pricing, i.e.,

$$P(t,T) = \sum_{i=1}^{T} C_T e^{-r_i \cdot (i-t)} + V_T e^{-r_T \cdot (T-t)}$$

  Where $P(t,T)$ is price, C_T is a coupon, V_T is the value of the bond (i.e., here \$100), and r_i is the interest rate of a $(0,i)$ bond.

- We then generalize to the more accurate pricing model given the data observed, the discrete case of pricing, i.e.,

$$P_T = \sum_{i=1}^{T} \frac{C_T}{(1+r_i)^i} + \frac{V_T}{(1+r_T)^T}$$

  Where the notation is the same as above.

- We then apply the Nelson-Siegel Model to derive a much nicer version of our yield and forward curves. The N-S model is described as follows:

$$y\left(m\right) = \beta_0 + \beta_1 \frac{\left[1 - \exp\left(-m/\tau\right)\right]}{m/\tau} + \beta_2 \left(\frac{\left[1 - \exp\left(-m/\tau\right)\right]}{m/\tau} - \exp\left(-m/\tau\right) \right)$$

  where $y\left(m\right)$ and m are yield and maturity respectively, and β_0, β_1, β_2 and τ, are parameters to be fitted via a least-squares. Explicitly:

  - β_0 is interpreted as the long run levels of interest rates (the loading is 1, it is a constant that does not decay)
  - β_1 is the short-term component (it starts at 1, and decays monotonically and quickly to 0);
  - β_2 is the medium-term component (it starts at 0, increases, then decays to zero);

– τ is the decay factor: small values produce slow decay and can better fit the curve at long maturities, while large values produce fast decay and can better fit the curve at short maturities; τ also governs where β_2 achieves its maximum.

For a complete history of the evolution of the code used, please visit my GitHub at:

https://github.com/jmostovoy/Mathematical_Finance

5.1 The Yield and Forward Curves

We begin by importing our bond data and transform it a little for ease of use. We now have either 44 or 43 data points for each date. Next, we create a function which automatically selects the bond associated to the closest date to $d + i$, where the date, $d \in \{2017\text{-}01\text{-}09, \ldots, 2017\text{-}01\text{-}20\} \setminus \{2017\text{-}01\text{-}14, 2017\text{-}01\text{-}15\}$ and $i = 1/2y, 1y, \ldots, 5y$ (y = years). This yields the bonds to be used as:

$$\begin{pmatrix} B_{d+1/2y} \\ \vdots \\ B_{d+5y} \end{pmatrix} = \begin{pmatrix} \text{2017-08-01} \\ \text{2018-02-01} \\ \text{2018-08-01} \\ \text{2019-02-01} \\ \text{2019-09-01} \\ \text{2020-03-01} \\ \text{2020-09-01} \\ \text{2021-03-01} \\ \text{2021-09-01} \\ \text{2022-03-01} \end{pmatrix} \quad \forall i$$

We then proceed to calculate the yield curves through "bootstrapping", which is an algorithm defined as follows:

1. Given d, First find $r_{1/2}$:= the yield rate on a zero coupon 6 month Government of Canada Bond. As an example, for the continuous pricing model, we do this by inputting values into the formula:

$$r_{1/2} = \frac{-\log\left(\frac{P(0,1/2)}{C_{1/2}+V_{1/2}}\right)}{1/2}$$

   As an explicit example, for d =2017-01-09, this is evaluated as:

$$r_{1/2} = \frac{-\log\left(\frac{100.38}{1.25/2+100}\right)}{1/2} \approx 0.4876\%$$

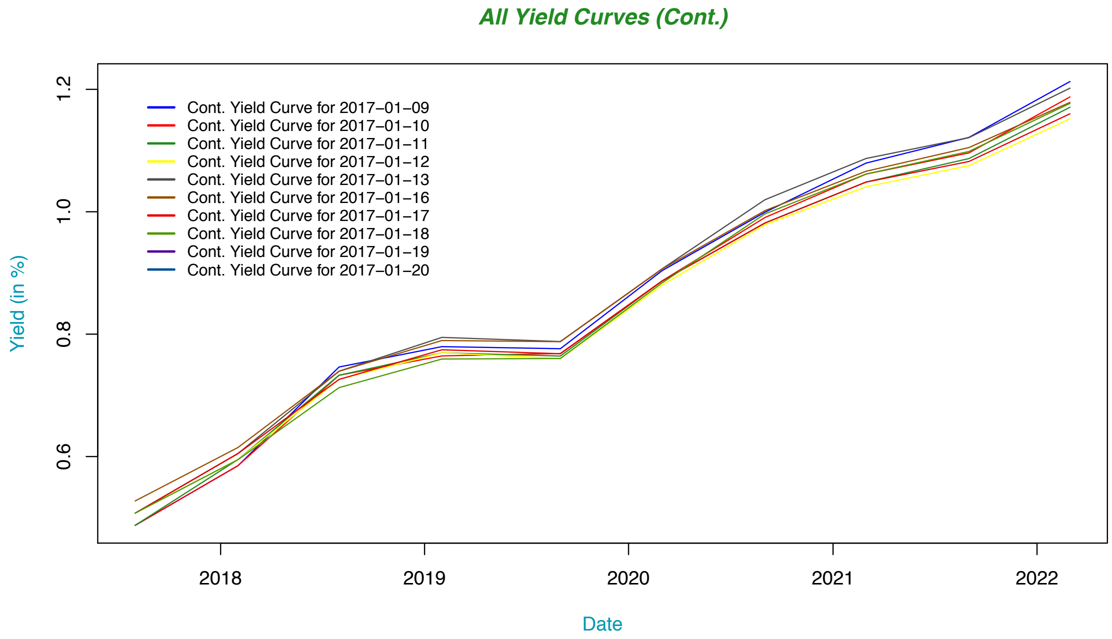
2. Next, we recursively solve for $r_1, r_{3/2}, \ldots, r_5$ by first obtaining r_k, $k < i$, then inputting this to solve with these values for r_i, i.e.:

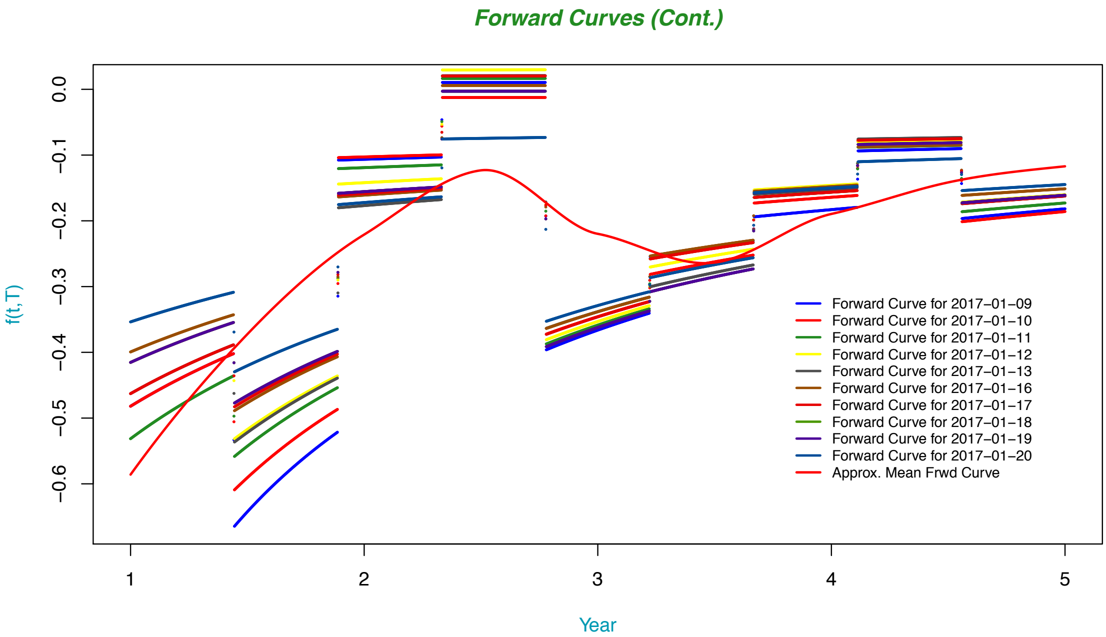
$$r_i = \frac{-\log\left(\frac{P(0,i) - \sum_{0<k<i} C_i e^{-k \cdot r_k}}{C_i + V_i}\right)}{i}$$

Please reference page 6, for a thorough explanation on how we derived the "Classical (Implied) Forward Rates (in %)". Thus, after performing the described (in Question 1) "bootstrapping", we report the results:

5.1.1 Under Continuous Assumptions

Yield Rates (in %)										
Date	$r_{1/2}$	r_1	$r_{3/2}$	r_2	$r_{5/2}$	r_3	$r_{7/2}$	r_4	$r_{9/2}$	r_5
2017-01-09	0.488	0.585	0.746	0.763	0.766	0.904	0.999	1.080	1.121	1.213
2017-01-10	0.488	0.585	0.733	0.764	0.768	0.887	0.991	1.062	1.096	1.188
2017-01-11	0.488	0.595	0.733	0.769	0.764	0.887	0.982	1.049	1.087	1.171
2017-01-12	0.507	0.595	0.726	0.769	0.760	0.880	0.979	1.041	1.075	1.152
2017-01-13	0.507	0.605	0.739	0.795	0.788	0.907	1.020	1.087	1.121	1.202
2017-01-16	0.527	0.615	0.739	0.790	0.788	0.907	1.002	1.066	1.105	1.179
2017-01-17	0.507	0.605	0.726	0.774	0.768	0.887	0.982	1.048	1.082	1.160
2017-01-18	0.507	0.595	0.713	0.759	0.760	0.884	0.996	1.062	1.098	1.177
2017-01-19	0.507	0.595	0.713	0.759	0.760	0.884	0.996	1.062	1.098	1.177
2017-01-20	0.527	0.605	0.713	0.764	0.788	0.904	1.011	1.077	1.126	1.198

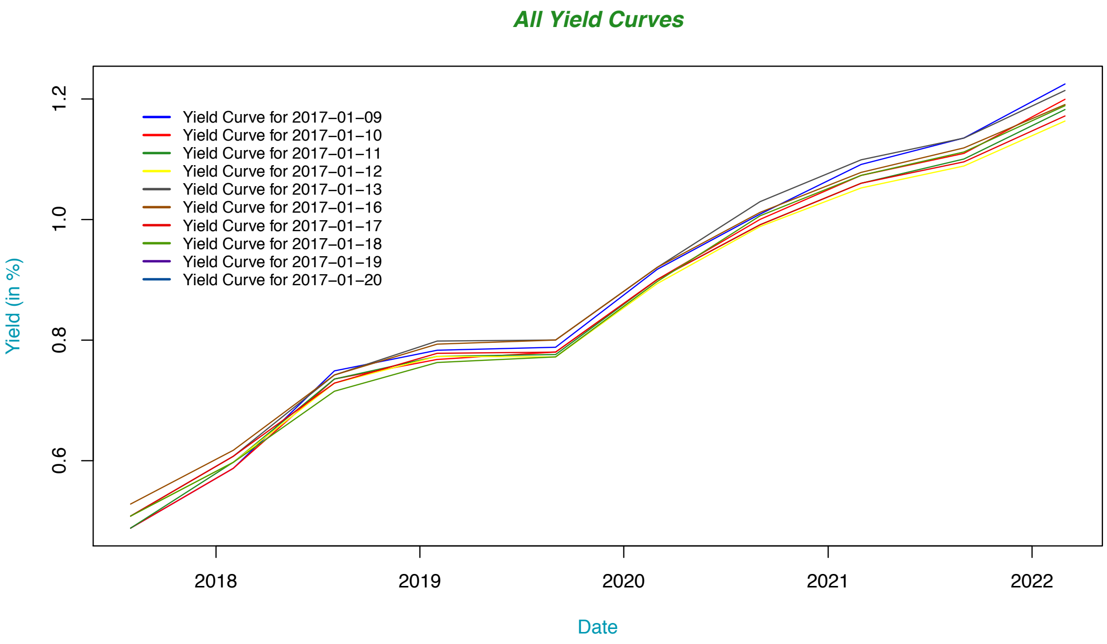


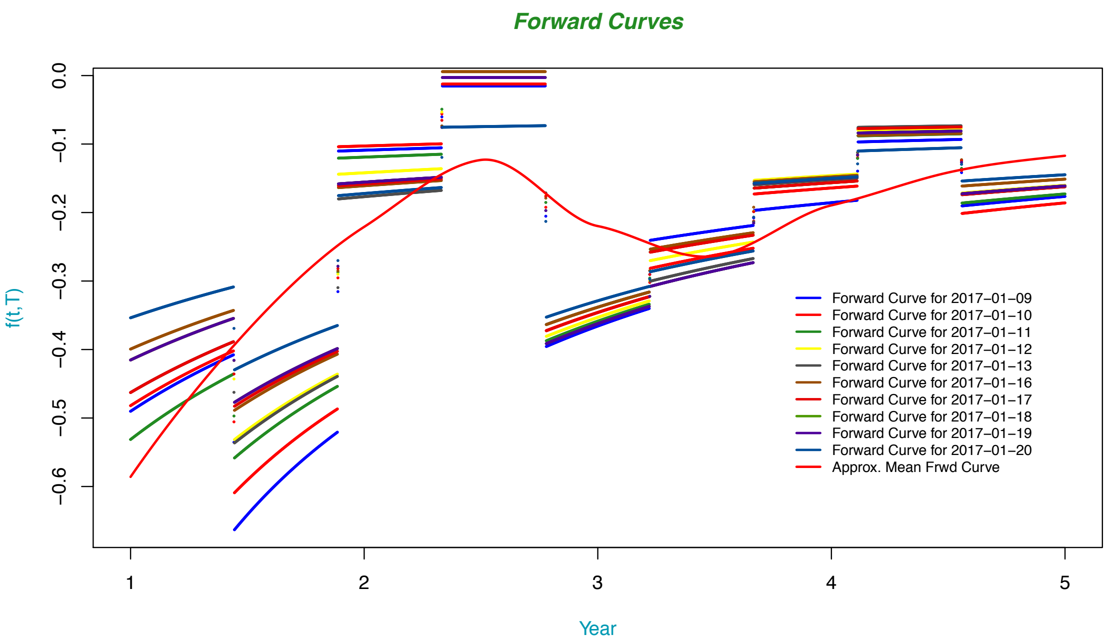


Classical (Implied) Forward Rates (in %)										
Date	f_{12}	f_{13}	f_{14}	f_{15}	f_{23}	f_{24}	f_{25}	f_{34}	f_{35}	f_{45}
2017-01-09	0.975	1.063	1.245	1.370	1.152	1.380	1.503	1.609	1.678	1.748
2017-01-10	0.944	1.038	1.221	1.339	1.133	1.360	1.471	1.587	1.640	1.694
2017-01-11	0.944	1.033	1.200	1.315	1.122	1.328	1.439	1.535	1.598	1.661
2017-01-12	0.944	1.023	1.190	1.292	1.102	1.313	1.408	1.524	1.561	1.597
2017-01-13	0.985	1.058	1.248	1.352	1.131	1.380	1.474	1.630	1.647	1.663
2017-01-16	0.965	1.053	1.217	1.320	1.141	1.344	1.439	1.547	1.589	1.630
2017-01-17	0.944	1.028	1.197	1.299	1.112	1.323	1.418	1.535	1.572	1.608
2017-01-18	0.924	1.028	1.218	1.323	1.133	1.365	1.457	1.597	1.619	1.640
2017-01-19	0.924	1.028	1.218	1.323	1.133	1.365	1.457	1.597	1.619	1.640
2017-01-20	0.924	1.053	1.235	1.347	1.183	1.390	1.488	1.598	1.641	1.684

5.1.2 Under Non-Continuous Assumptions

Yield Rates (in %)										
Date	$r_{1/2}$	r_1	$r_{3/2}$	r_2	$r_{5/2}$	r_3	$r_{7/2}$	r_4	$r_{9/2}$	r_5
2017-01-09	0.488	0.587	0.749	0.783	0.788	0.918	1.009	1.091	1.135	1.225
2017-01-10	0.488	0.587	0.736	0.768	0.780	0.901	1.000	1.073	1.110	1.199
2017-01-11	0.488	0.597	0.735	0.773	0.776	0.901	0.991	1.060	1.100	1.182
2017-01-12	0.508	0.597	0.729	0.773	0.772	0.894	0.988	1.052	1.089	1.163
2017-01-13	0.508	0.607	0.742	0.798	0.800	0.921	1.030	1.099	1.135	1.214
2017-01-16	0.528	0.617	0.742	0.793	0.800	0.921	1.012	1.078	1.119	1.191
2017-01-17	0.508	0.607	0.729	0.778	0.780	0.901	0.991	1.060	1.096	1.172
2017-01-18	0.508	0.597	0.715	0.763	0.772	0.897	1.006	1.073	1.112	1.189
2017-01-19	0.508	0.597	0.715	0.763	0.772	0.897	1.006	1.073	1.112	1.189
2017-01-20	0.528	0.607	0.715	0.768	0.801	0.918	1.021	1.089	1.140	1.210


All Yield Curves
Yield Curve for 2017−01−09
Yield Curve for 2017−01−10
Yield Curve for 2017−01−11
Yield Curve for 2017−01−12
Yield Curve for 2017−01−13
Yield Curve for 2017−01−16
Yield Curve for 2017−01−17
Yield Curve for 2017−01−18
Yield Curve for 2017−01−19
Yield Curve for 2017−01−20
Yield (in %)
0.6
0.8
1.0
1.2
2018
2019
2020
2021
2022
Date


Forward Curves
Forward Curve for 2017−01−09
Forward Curve for 2017−01−10
Forward Curve for 2017−01−11
Forward Curve for 2017−01−12
Forward Curve for 2017−01−13
Forward Curve for 2017−01−16
Forward Curve for 2017−01−17
Forward Curve for 2017−01−18
Forward Curve for 2017−01−19
Forward Curve for 2017−01−20
Approx. Mean Frwd Curve
f(t, T)
0.0
−0.1
−0.2
−0.3
−0.4
−0.5
−0.6
1
2
3
4
5
Year

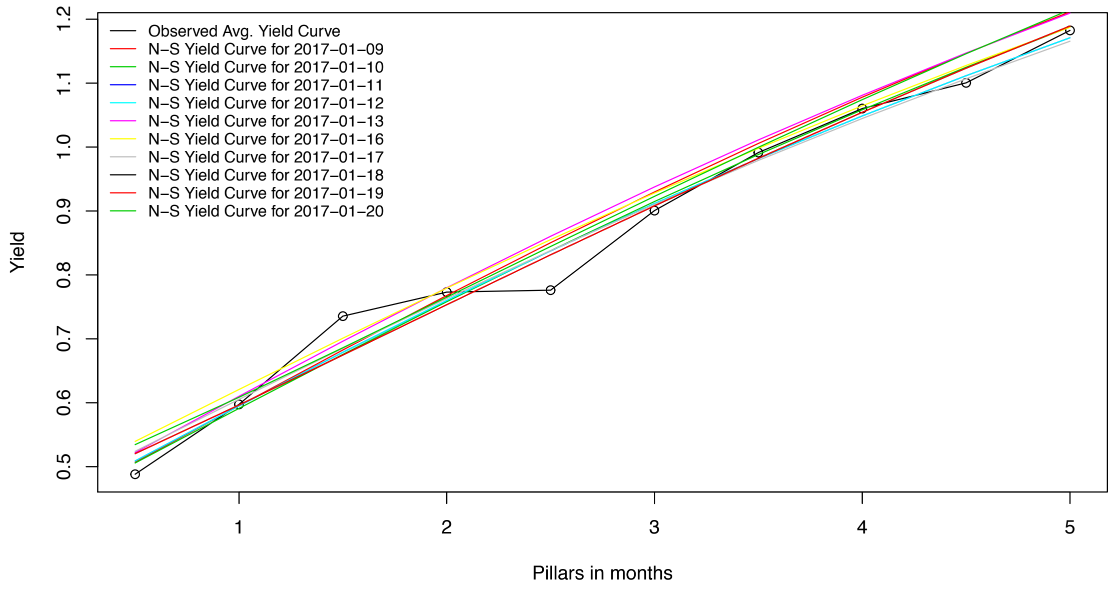
Classical (Implied) Forward Rates (in %)										
Date	f_{12}	f_{13}	f_{14}	f_{15}	f_{23}	f_{24}	f_{25}	f_{34}	f_{35}	f_{45}
2017-01-09	0.979	1.083	1.260	1.385	1.187	1.401	1.520	1.615	1.688	1.760
2017-01-10	0.949	1.058	1.236	1.353	1.166	1.379	1.488	1.593	1.649	1.706
2017-01-11	0.949	1.053	1.215	1.329	1.156	1.348	1.456	1.541	1.607	1.673
2017-01-12	0.949	1.042	1.205	1.305	1.136	1.333	1.424	1.530	1.569	1.609
2017-01-13	0.990	1.078	1.264	1.366	1.166	1.401	1.492	1.636	1.656	1.675
2017-01-16	0.970	1.073	1.232	1.335	1.176	1.364	1.457	1.552	1.597	1.642
2017-01-17	0.949	1.047	1.212	1.313	1.146	1.343	1.435	1.541	1.580	1.620
2017-01-18	0.929	1.048	1.232	1.337	1.167	1.385	1.474	1.603	1.628	1.652
2017-01-19	0.929	1.048	1.232	1.337	1.167	1.385	1.474	1.603	1.628	1.652
2017-01-20	0.929	1.073	1.250	1.361	1.217	1.411	1.506	1.604	1.650	1.696

5.1.3 Under Nelson-Siegel Model

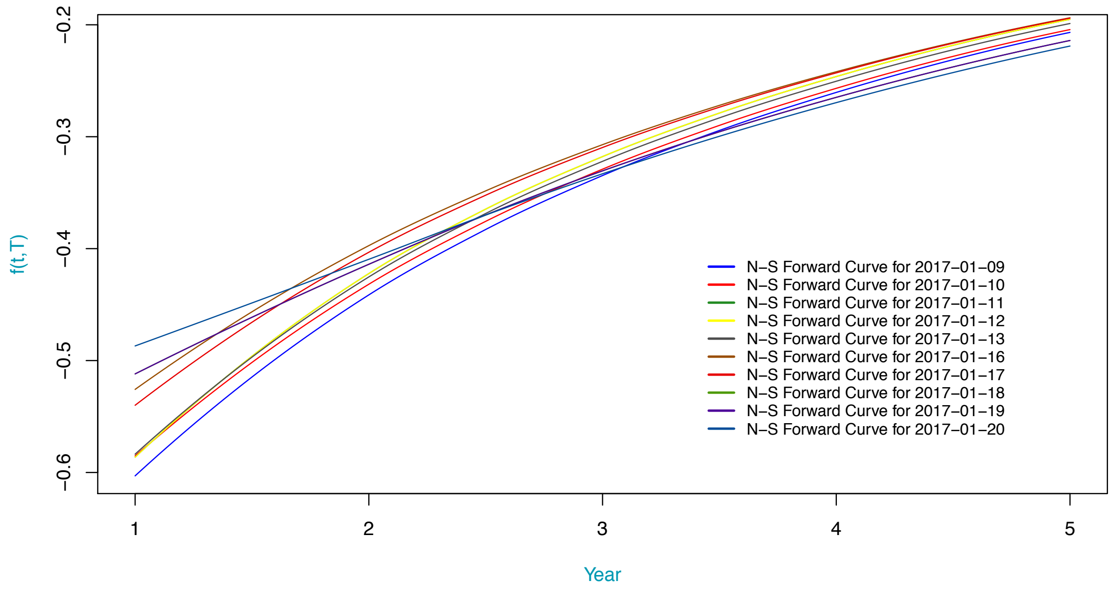
We report (after using our Yield Rates from section 1.2.2 as the necessary inputs):

Yield Rates (in %)										
Date	$r_{1/2}$	r_1	$r_{3/2}$	r_2	$r_{5/2}$	r_3	$r_{7/2}$	r_4	$r_{9/2}$	r_5
2017-01-09	0.507	0.595	0.683	0.768	0.850	0.930	1.006	1.078	1.147	1.212
2017-01-10	0.506	0.591	0.676	0.758	0.838	0.915	0.988	1.058	1.125	1.188
2017-01-11	0.509	0.595	0.679	0.760	0.838	0.912	0.982	1.049	1.112	1.171
2017-01-12	0.509	0.595	0.679	0.760	0.838	0.912	0.982	1.049	1.112	1.171
2017-01-13	0.522	0.610	0.696	0.780	0.860	0.938	1.011	1.081	1.147	1.210
2017-01-16	0.540	0.621	0.701	0.779	0.855	0.928	0.998	1.065	1.128	1.187
2017-01-17	0.524	0.605	0.685	0.763	0.838	0.910	0.979	1.045	1.107	1.166
2017-01-18	0.520	0.596	0.675	0.753	0.831	0.908	0.982	1.054	1.123	1.189
2017-01-19	0.520	0.596	0.675	0.753	0.831	0.908	0.982	1.054	1.123	1.189
2017-01-20	0.534	0.609	0.686	0.765	0.844	0.923	1.000	1.074	1.146	1.215

Fitted Nelson–Siegel Yield Curves



N–S Forward Curves



5.2 Covariance Matrix For Relevant Time Series

We now turn to calculating the covariance matrix of the time series of daily log-returns of yield rates, forward rates and forward curve. Explicitly, we consider the random variable X_i, which has a time series $X_{(i,j)}$ given by:

$$X_{(i,j)} = \log\left(\frac{r_{(i,j+1)}}{r_{(i,j)}}\right), j \in J$$

We define our covariance matrix for the yield as follows:

$$\hat{\Sigma} = \begin{pmatrix} \sigma_{X_1,X_1} & \cdots & \sigma_{X_1,X_5} \\ \vdots & \ddots & \vdots \\ \sigma_{X_5,X_1} & \cdots & \sigma_{X_5,X_5} \end{pmatrix}$$

For the forward rates, we define the implied yield rate from time $\hat{\imath}$, $1 \leq \hat{\imath} \leq 4$ to time $\hat{\jmath}$, $\hat{\imath} \leq \hat{\jmath} \leq 5$ as $f_{\hat{\imath}\hat{\jmath}}$. As such, we have 10 possibilities for $\hat{\imath}\hat{\jmath}$, $\hat{\imath}\hat{\jmath} \in \{12, 13, 14, 15, 23, 24, 25, 34, 35, 45\}$

Thus, the analogue for the described above time series would be $Y_{(i,j)}$, where $j = d$ (as described in Question 1) and $j = \hat{\imath}\hat{\jmath}$. Thus, $\hat{\Sigma}$ will now be $\in M_{10\times10}(\mathbb{R})$.

Another analogue for the forward rate time series will be the values, Z_{ij}, where i, j are analogous to the how was described for the yield curves, except Z_{ij} is the value at ij on the forward curve. We call these two different approaches $forward1$ and $forward2$ respectively.

5.2.1 Under Continuous Assumptions

$$\hat{\Sigma}_{yield} = 10^{-4} \cdot \begin{pmatrix} 1.8945 & 1.7087 & 1.5952 & 0.6403 & 0.6333 \\ 1.7087 & 2.8460 & 2.4633 & 2.4317 & 2.4154 \\ 1.5952 & 2.4633 & 2.8671 & 2.8959 & 3.1087 \\ 0.6403 & 2.4317 & 2.8959 & 4.2389 & 4.4687 \\ 0.6333 & 2.4154 & 3.1087 & 4.4687 & 4.8311 \end{pmatrix}$$

$\hat{\Sigma}_{forward1} =$

$$10^{-4} \cdot \begin{pmatrix} 4.567 & 2.090 & 1.867 & 2.119 & -0.391 & 0.511 & 1.299 & 1.418 & 2.148 & 2.879 \\ 2.090 & 2.182 & 2.490 & 3.024 & 2.274 & 2.690 & 3.337 & 3.108 & 3.871 & 4.634 \\ 1.867 & 2.490 & 3.867 & 4.528 & 3.114 & 4.871 & 5.420 & 6.635 & 6.579 & 6.523 \\ 2.119 & 3.024 & 4.528 & 5.832 & 3.931 & 5.738 & 7.076 & 7.553 & 8.657 & 9.762 \\ -0.391 & 2.274 & 3.114 & 3.931 & 4.945 & 4.874 & 5.379 & 4.802 & 5.597 & 6.393 \\ 0.511 & 2.690 & 4.871 & 5.738 & 4.874 & 7.059 & 7.489 & 9.254 & 8.803 & 8.352 \\ 1.299 & 3.337 & 5.420 & 7.076 & 5.379 & 7.489 & 9.012 & 9.609 & 10.837 & 12.067 \\ 1.418 & 3.108 & 6.635 & 7.553 & 4.802 & 9.254 & 9.609 & 13.727 & 12.024 & 10.320 \\ 2.148 & 3.871 & 6.579 & 8.657 & 5.597 & 8.803 & 10.837 & 12.024 & 13.470 & 14.919 \\ 2.879 & 4.634 & 6.523 & 9.762 & 6.393 & 8.352 & 12.067 & 10.320 & 14.919 & 19.522 \end{pmatrix}$$

$$\hat{\Sigma}_{forward2} = \begin{pmatrix} 0.015 & -0.012 & -0.001 & 0.008 & -0.003 \\ -0.012 & 0.023 & -0.019 & 0.008 & -0.004 \\ -0.001 & -0.019 & 0.031 & -0.024 & 0.009 \\ 0.008 & 0.008 & -0.024 & 0.043 & -0.030 \\ -0.003 & -0.004 & 0.009 & -0.030 & 0.060 \end{pmatrix}$$

5.2.2 Under Non-Continuous Assumptions

$$\hat{\Sigma}_{yield} = 10^{-4} \cdot \begin{pmatrix} 1.9008 & 1.7136 & 1.6044 & 0.6508 & 0.6410 \\ 1.7136 & 2.8500 & 2.4649 & 2.4358 & 2.4212 \\ 1.6044 & 2.4649 & 2.8584 & 2.8845 & 3.0968 \\ 0.6508 & 2.4358 & 2.8845 & 4.2222 & 4.4579 \\ 0.6410 & 2.4212 & 3.0968 & 4.4579 & 4.8266 \end{pmatrix}$$

$$\hat{\Sigma}_{forward1} = 10^{-4} \cdot \begin{pmatrix} 4.772 & 3.386 & 3.860 & 3.739 & 2.226 & 3.546 & 3.517 & 4.505 & 3.978 & 3.459 \\ 3.386 & 3.694 & 3.847 & 4.052 & 3.909 & 4.003 & 4.194 & 4.055 & 4.290 & 4.500 \\ 3.860 & 3.847 & 5.523 & 5.610 & 3.795 & 6.101 & 5.989 & 7.788 & 6.776 & 5.792 \\ 3.739 & 4.052 & 5.610 & 5.846 & 4.264 & 6.259 & 6.303 & 7.715 & 7.033 & 6.361 \\ 2.226 & 3.909 & 3.795 & 4.264 & 5.241 & 4.333 & 4.701 & 3.643 & 4.498 & 5.302 \\ 3.546 & 4.003 & 6.101 & 6.259 & 4.333 & 6.989 & 6.847 & 8.934 & 7.750 & 6.601 \\ 3.517 & 4.194 & 5.989 & 6.303 & 4.701 & 6.847 & 6.906 & 8.414 & 7.697 & 6.990 \\ 4.505 & 4.055 & 7.788 & 7.715 & 3.643 & 8.934 & 8.414 & 12.827 & 10.134 & 7.542 \\ 3.978 & 4.290 & 6.776 & 7.033 & 4.498 & 7.750 & 7.697 & 10.134 & 8.847 & 7.593 \\ 3.459 & 4.500 & 5.792 & 6.361 & 5.302 & 6.601 & 6.990 & 7.542 & 7.593 & 7.615 \end{pmatrix}$$

5.3 Eigenvalues and Eigenvectors

We know calculate the eigenvalues and eigenvectors of both covariance matrices.

To do so, we define our observed eigenvalues and eigenvectors respectively as:

$$\hat{\Lambda} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{bmatrix}, \qquad \hat{E}_i = \begin{bmatrix} e_{i1} \\ e_{i2} \\ e_{i3} \\ e_{i4} \\ e_{i5} \end{bmatrix}$$

5.3.1 Under Continuous Assumptions

$$\hat{\Lambda}_{yield} = 10^{-3} \cdot \begin{bmatrix} 1.3304 \\ 0.2876 \\ 0.0409 \\ 0.0061 \\ 0.0027 \end{bmatrix} \quad \hat{\Lambda}_{forward1} = 10^{-5} \cdot \begin{bmatrix} 5588.2442 \\ 55.5107 \\ 32.5211 \\ 6.3227 \\ 0.0013 \\ 0.0002 \\ 0.0001 \\ \approx 0 \\ \approx 0 \\ \approx 0 \end{bmatrix} \quad \hat{\Lambda}_{forward2} = 10^{-4} \cdot \begin{bmatrix} 941.9447 \\ 432.8190 \\ 269.9097 \\ 67.0916 \\ 0.0000 \end{bmatrix}$$

$$\hat{E}_{yield} = \left[\hat{E}_1, \ldots, \hat{E}_5\right] = \begin{bmatrix} -0.1823 & -0.6917 & -0.3672 & 0.5799 & 0.1312 \\ -0.3914 & -0.4430 & 0.7666 & -0.2002 & 0.1514 \\ -0.4406 & -0.2417 & -0.4632 & -0.6412 & -0.3489 \\ -0.5406 & 0.3326 & 0.1551 & 0.4609 & -0.6005 \\ -0.5720 & 0.3954 & -0.1973 & 0.0105 & 0.6910 \end{bmatrix}$$

$$\left[\hat{E}_1, \ldots, \hat{E}_5\right]_{forward1} = \begin{bmatrix} -0.196 & 0.130 & 0.860 & 0.159 & 0.386 & 0.094 & 0.054 & -0.134 & 0.000 & 0.031 \\ -0.212 & 0.390 & 0.245 & -0.179 & -0.797 & -0.029 & -0.022 & -0.271 & 0.007 & 0.033 \\ -0.304 & -0.033 & 0.145 & -0.201 & -0.072 & -0.251 & 0.065 & 0.642 & 0.394 & -0.453 \\ -0.315 & 0.058 & 0.023 & 0.076 & -0.048 & -0.081 & -0.349 & 0.510 & -0.589 & 0.393 \\ -0.222 & 0.599 & -0.258 & -0.449 & 0.412 & 0.156 & -0.246 & -0.137 & -0.063 & -0.216 \\ -0.341 & -0.091 & -0.102 & -0.322 & 0.074 & 0.035 & 0.810 & -0.013 & -0.234 & 0.209 \\ -0.340 & 0.041 & -0.157 & 0.062 & 0.125 & -0.223 & -0.122 & -0.100 & 0.611 & 0.625 \\ -0.428 & -0.604 & 0.012 & -0.226 & -0.079 & 0.500 & -0.316 & -0.183 & 0.038 & -0.102 \\ -0.383 & -0.161 & -0.121 & 0.249 & 0.077 & -0.635 & -0.065 & -0.407 & -0.241 & -0.337 \\ -0.338 & 0.260 & -0.249 & 0.689 & -0.073 & 0.438 & 0.188 & 0.092 & 0.076 & -0.185 \end{bmatrix}$$

$$\hat{E}_{forward2} = \left[\hat{E}_1, \ldots, \hat{E}_5\right] = \begin{bmatrix} 0.0637 & -0.2707 & 0.6028 & -0.4669 & -0.5842 \\ 0.1987 & 0.5796 & -0.4077 & 0.0116 & -0.6769 \\ -0.3870 & -0.5550 & -0.2209 & 0.5424 & -0.4465 \\ 0.6086 & 0.0598 & 0.4439 & 0.6544 & -0.0264 \\ -0.6606 & 0.5284 & 0.4738 & 0.2437 & -0.0227 \end{bmatrix}$$

5.3.2 Under Non-Continuous Assumptions

$$\hat{\Lambda}_{yield} = 10^{-3} \cdot \begin{bmatrix} 1.3289 \\ 0.2872 \\ 0.0409 \\ 0.0060 \\ 0.0027 \end{bmatrix} \qquad \hat{\Lambda}_{forward1} = 10^{-5} \cdot \begin{bmatrix} 588.2442 \\ 55.5107 \\ 32.5211 \\ 6.3227 \\ 0.0013 \\ 0.0002 \\ 0.0001 \\ \approx 0 \\ \approx 0 \\ \approx 0 \end{bmatrix}$$

$$\hat{E}_{yield} = \left[\hat{E}_1, \ldots, \hat{E}_5\right] = \begin{bmatrix} -0.1841 & -0.6917 & -0.3678 & 0.5786 & 0.1325 \\ -0.3927 & -0.4402 & 0.7683 & -0.1974 & 0.1509 \\ -0.4401 & -0.2426 & -0.4611 & -0.6436 & -0.3472 \\ -0.5398 & 0.3327 & 0.1514 & 0.4603 & -0.6026 \\ -0.5717 & 0.3977 & -0.1972 & 0.0102 & 0.6899 \end{bmatrix}$$

$$\left[\hat{E}_1, \ldots, \hat{E}_5\right]_{forward1} = \begin{bmatrix} -0.196 & 0.130 & 0.860 & 0.159 & 0.386 & 0.094 & 0.054 & -0.134 & 0.000 & 0.031 \\ -0.212 & 0.390 & 0.245 & -0.179 & -0.797 & -0.029 & -0.022 & -0.271 & 0.007 & 0.033 \\ -0.304 & -0.033 & 0.145 & -0.201 & -0.072 & -0.251 & 0.065 & 0.642 & 0.394 & -0.453 \\ -0.315 & 0.058 & 0.023 & 0.076 & -0.048 & -0.081 & -0.349 & 0.510 & -0.589 & 0.393 \\ -0.222 & 0.599 & -0.258 & -0.449 & 0.412 & 0.156 & -0.246 & -0.137 & -0.063 & -0.216 \\ -0.341 & -0.091 & -0.102 & -0.322 & 0.074 & 0.035 & 0.810 & -0.013 & -0.234 & 0.209 \\ -0.340 & 0.041 & -0.157 & 0.062 & 0.125 & -0.223 & -0.122 & -0.100 & 0.611 & 0.625 \\ -0.428 & -0.604 & 0.012 & -0.226 & -0.079 & 0.500 & -0.316 & -0.183 & 0.038 & -0.102 \\ -0.383 & -0.161 & -0.121 & 0.249 & 0.077 & -0.635 & -0.065 & -0.407 & -0.241 & -0.337 \\ -0.338 & 0.260 & -0.249 & 0.689 & -0.073 & 0.438 & 0.188 & 0.092 & 0.076 & -0.185 \end{bmatrix}$$

Due to almost exact similarity to $\left[\hat{E}_1, \ldots, \hat{E}_5\right]_{forward2}$ in the continuous case, we omit $\left[\hat{E}_1, \ldots, \hat{E}_5\right]_{forward2}$ for the discrete case.

Second Assignment:

The Question

Calculate the 1, 3 and 5 year probability of default of Bell Canada under each of the following assumptions:

1. A Markov chain model with two states: solvency and default, calibrated to one of its bond prices.
2. A Merton model, using company's assets, company's liabilities and asset volatilities implied by their stock volatility

Note: The assignment must contain: all explanations of the work done, all assumptions made, and the results with probabilities in table and chart formats.

6 Markov Chain Model

6.1 Necessary Theory

6.1.1 Default Rates

We recall that the discounted value of cash flows over a countable period of times $t \in I$, when $\exists$ a probability of default, is given by:

$$V = \sum_{i=1}^{n} p_i e^{-r_i t_i} q_i$$

Where:
V := the present valuation of our security.
n := the number of payments
p_i := the amount paid at t_i.
r_i := is the continuously compounded (risk free) interest rate at t_i.
q_i := is the probability of solvency at t_i.

Thus, if we let $h_i := \frac{-\log(q_i)}{t_i}$, then:

$$V = \sum_{i \in I} p_i e^{-r_i t_i} q_i = \sum_{i \in I} p_i e^{-(r_i + h_i)t_i}$$

Furthermore, if we assume a two credit states model (Solvency or Default), and assume the following:

$$\mathbb{P}(\text{Solvent at } t_{i+1} \mid \text{Solvent at } t_i) = q_i$$

Then if s := solvent, d := default, and if $S := \{s, d\}$ is our state space, we have that:

$$\begin{aligned}\mathbb{P}(X_{i+2} = s | X_i = s) &= \sum_{j \in S} \mathbb{P}\big(X_{i+2} | X_{i+1} = j, X_i = s\big)\mathbb{P}(X_{i+1} = j | X_i = s\big) \\ &= \mathbb{P}\big(X_{i+2} | X_{i+1} = s, X_i = s\big)\mathbb{P}(X_{i+1} = s | X_i = s) \\ &= q_i^2\end{aligned}$$

Thus, by induction and if we let $\mathbb{P}(X_1|X_0) := q$, then $\mathbb{P}(X_{i+k}|X_i = s) = q^k \ \forall i \in \{1, \dots, n\}$. And furthermore:

$$q_i = q^{t_i} \implies h_i = \frac{-\log\left(q^{t_i}\right)}{t_i} = -\log(q)$$

Moreover, we recognize this setup as a Markov Chain with the transition matrix of:

$$p_{ij} = P = \begin{pmatrix} q & 1-q \\ 0 & 1 \end{pmatrix}$$

6.1.2 Setting up the Notation for Yield

As we mentioned in the previous section, given a corporate bond, the value of this bond is $V = \sum_{i \in I} p_i e^{-(r_i + h_i)t_i}$. Thus, if we would like to solve for the default rate, $1 - q = 1 - e^{-h_i}$, then we unfortunately have two unknown sequences of numbers, i.e., r_i and h_i. Thus, to be able to solve for the h_i's, we need to first solve for the r_i's. However, since we are making the additional assumption that $h_i = -\log(q^{t_i})/t_i = -\log(q)\forall i$, then if we are interested in finding the $1, 2, 3, \dots$ probability of default, then we need only compute the probability of default for year one: e^{-h_1}, and by definition, $\mathbb{P}(\text{Default for } t \in (0, t], t \in \mathbb{N}) = e^{-h_1 \cdot t}$.

For example; assume company Z has a n bonds, each pays a semi-annual coupon. We denote these bonds by $Z_1, Z_2, \dots, Z_n$. Furthermore, assume today's date is d, and that bond Z_i matures on $t(Z)_i$, where if $y := 1$ year, $t(Z)_i \in (d + (i-1)y/2, d + iy/2)$.

Next, let us assume we have the same set up for 3 government bonds, $R_1, \dots, R_3$, and that $t(R)_1, \dots, t(R)_3$ satisfy the same constraints as for $t(Z)_i$ (but $t(R)_i$ need not equal $t(Z)_i$).

We also let $P(Z)_i$, and $P(R)_i$ denote the prices for Z's and the government bond's respectively, which are given. The annual coupon payment for Z and the government is denoted $C(Z)_i$ and $C(R)_i$ respectively. And finally, the face value is denoted $F(Z)_i$ and $F(R)_i$ for each.

We now introduce one more necessary component. We let $\delta(d, t_j) : (\text{dates})^2 \to \{0, \dots, 364\} \cup \mathbb{1}_{\text{leap year}}\{365\}$, so that: $\delta(d, t_j) = \textit{days\_between}(d, t_j)$

We now have the necessary notation and theory to solve for Z's probability of default implicitly by computing the r_1 and ρ_1 - the 1 year yield of R and Z respectively. To do so, we recall that for a government bond:

$$P(R)_i = -C(R)_i \cdot \underbrace{\left(\delta\Big(d, \inf\Big\{\{t|t = d + n/2y, n \in \mathbb{N}\} \cap \{t|t \geq t(R)_i\}\Big\}\Big) - \delta(d, t(R)_i)\right)}_{\text{Days since last coupon payment}} \Big/ \left(365 + \mathbb{1}_{\text{leap year}}\right)$$
$$+ \left(\sum_{j=1}^{i-1} \frac{C(R)_i}{2} \cdot \exp\left(-r_{j/2} \cdot (j/2)\right)\right) + \left(\frac{C(R)_i}{2} + F(R)_i\right) \cdot \exp\left(-r_{\delta(d,t(R)_i)} \cdot \delta(d, t(R)_i)\right)$$

And letting $R = Z$, $r_j = \rho_j$, we get our equation for pricing a corporate bond as well.

To simplify our notation, let us introduce the two functions:
$\psi(d, t(X)_i) :=$ Days since last coupon payment (explicitly explained above)
$\phi(m) = m/(365 + \mathbb{1}_{\text{leap year}}), \quad \phi : \text{Im}(\delta) \to \mathbb{Q}^+$

We can now explicitly solve for $r_{k/2}, k \in \{1, \dots, n-1\}$. We do this by utilizing a recursive algorithm (called bootstrapping) which we introduce in the next section.

6.1.3 Bootstrapping for the Yields

We consider the same set up as above for a government bond (and note the complete analogous for the yield of a corporate bond) as above. After moving a few variables to the other side, taking a log and the dividing by $-\delta(d, t(R)_i)$, we can see:

$$r_{\delta(d,t(R)_i)} = \frac{-1}{\phi(\delta(d, t(R)_i))} \log \left(\frac{P(R)_i + C(R)_i \cdot \psi(d, t(X)_i) - (C(R)_i/2) \cdot \sum_{j=1}^{i-1} \exp\left(-r_{j/2} \cdot (j/2)\right)}{C(R)_i/2 + F(R)_i} \right)$$

Therefore, what seems natural is a "bootstrapping algorithm", that is, given $r_{1/2}, r_1, \ldots, r_k$, we can easily calculate r_j, where $j \in (0, k + 1/2y)$ from the above equation. This method of first starting with calculating r_j, $0 < j < 1/2$, then using that to make an assumption about $r_{1/2}$, then using that assumed value to calculate $r_k, 1/2 \leq k < 1$, and so on is how we will solve for the necessary information for calculating default rates.

6.1.4 Inducing the Probability of Default from the Calculated Yields

Once we have calculated $r_{1/2}$ and r_1 from the above algorithm, we then perform the exact same operation to calculate $\rho_{1/2}$ and ρ_1. From here, the theory suggests that:

$$\rho_1 = h_1 + r_1 \implies \mathbb{P}(\text{Solvent for } t \in (0, 1]) = q = e^{-(h_1)} = e^{-(\rho_1 - r_1)}$$

6.2 The Theory Applied to the Question at Hand

We report our data set as follows:

$$d = \text{2017-04-04}$$

Bond	Eff. Maturity ($t(R)_i$)	Price ($P(R)_i$)	Coupon ($C(R)_i$)
R_1	2017-05-01	99.98	0.25
R_1'	2017-09-01	100.38	1.50
R_2	2017-11-01	99.81	0.25
R_2'	2018-02-01	100.53	1.25
R_2''	2018-03-01	100.56	1.25
R_3	2018-05-01	99.57	0.25
R_3'	2018-06-01	104.14	4.25
R_3''	2018-08-01	99.77	0.50
R_3'''	2018-09-01	100.80	1.25

Bond	Eff. Maturity ($t(Z)_i$)	Price ($P(Z)_i$)	Coupon ($C(Z)_i$)
Z_1	2017-09-13	101.37	4.37
Z_2	2018-03-16	102.94	4.40
Z_2'	2018-04-26	103.73	4.88
Z_3	2018-09-10	103.00	3.50
Z_3'	2019-02-26	107.41	5.52

We begin with the first data set (Government bond data). For the first four entries, no coupon is paid in between today's data and the maturity date; therefore, we can immediately solve for $r_{\delta(d,t(R)_i}$, $i = 1, 2$ by using the formulas previously derived and discussed in the preceding sections:

$$r_{\phi(\delta(d,\text{2017-05-01}))} = r_{27/365} = -\left(\frac{365}{27}\right) \cdot \log\left(\frac{99.98 + (.25)(154/365) - (.25/2)(0)}{(.25/2) + 100}\right) = 0.5337\%$$

$$r_{\phi(\delta(d,\text{2017-09-01}))} = r_{150/365} = -\left(\frac{365}{150}\right) \cdot \log\left(\frac{100.38 + (1.50)(34/365) - (1.50/2)(0)}{(1.50/2) + 100}\right) = 0.5568\%$$

Thus, to find $r_{1/2}$ we linearly extrapolate between $r_{27/365}$ and $r_{150/365}$. We see that we get an intercept of $b = 5.286266 \cdot 10^{-3}$, and a slope of $m = 1.878135 \cdot 10^{-6}$ (per day). And thus we see that:

$$r_{1/2} = 5.286266 \cdot 10^{-3} + 1.878135 \cdot 10^{-6}(183) = 0.5630\%$$

Now, we can use $r_{1/2}$ to find r_x, $x \in (1/2, 1)$:

$$r_{\phi(\delta(d,\text{2017-11-01}))} = r_{211/365} = -\left(\frac{365}{211}\right) \log\left(\frac{99.81 + (.25)(154/365) - (.25/2)(e^{-(0.005630)(1/2)})}{(.25/2) + 100}\right)$$
$$= 0.5783\%$$

$$r_{\phi(\delta(d,\text{2018-02-01}))} = r_{303/365} = -\left(\frac{365}{303}\right) \log\left(\frac{100.53 + (1.25)(62/365) - (1.25/2)(e^{-(0.005630)(1/2)})}{(1.25/2) + 100}\right)$$
$$= 0.6072\%$$

$$r_{\phi(\delta(d,\text{2018-03-01}))} = r_{331/365} = -\left(\frac{365}{331}\right) \log\left(\frac{100.56 + (1.25)(34/365) - (1.25/2)(e^{-(0.005630)(1/2)})}{(1.25/2) + 100}\right)$$
$$= 0.6284\%$$

Thus, to find r_1 we fit the best linear model according to $r_{211/365}$, $r_{303/365}$, and $r_{331/365}$. We see that we get an intercept of $b = 4.943038 \cdot 10^{-3}$, and a slope of $m = 3.917085 \cdot 10^{-6}$ (per day). And thus we see that:

$$r_1 = 4.943038 \cdot 10^{-3} + 3.917085 \cdot 10^{-6}(365) = 0.6373\%$$

Now, we perform the same analysis on Bell's (Z's) bond data as follows:

$$\rho_{\phi(\delta(d,\text{2017-09-13}))} = \rho_{162/365} = -\left(\frac{365}{162}\right) \cdot \log\left(\frac{101.37 + (4.37)(22/365) - (4.37/2)(0)}{(4.37/2) + 100}\right) = 1.2195\%$$

Now, since $\rho_{162/365}$ is 21 days away from $\rho_{1/2}$, and since we only have 1 data point between now and 6 months from now, we make the assumption that the linear extrapolation we carried out to get $r_{1/2}$ for R implies the linear extrapolation formula for $\rho_x, x \in (0, 1/2]$ takes the form: $\rho_{days/365} = b' + m' = b' + m \cdot (\rho_{days/365}/r_{days/365})(days)$, where m is from the previous section ($m = 1.878135 \cdot 10^{-6}$). I.e., m' equal to the previous m times the proportion between the two yield rates. Thus, plugging in $days = 162 \implies \rho_{162/365} = 1.2195\%$, $r_{162/365} = 0.5591\% \implies m' = 4.0965581 \cdot 10^{-6}$ $b' = 1.15314 \cdot 10^{-2}$, and hence we now have:

$$\rho_{1/2} = 1.153171784 \cdot 10^{-2} + 4.0965581 \cdot 10^{-6}(183) = 1.2281\%$$

We now use this value to solve for $\rho_v, v \in (1/2, 1)$:

$$\rho_{\phi(\delta(d,\text{2018-03-16}))} = \rho_{346/365} = -\left(\frac{365}{346}\right)\log\left(\frac{102.94 + (4.40)(19/365) - (4.40/2)(e^{-(0.012195)(1/2)})}{(4.40/2) + 100}\right)$$
$$= 1.2643\%$$

And in doing the same trick as we did to find $\rho_{1/2}$, we see that our linear model should be $\rho_{days/365} = b' + \frac{0.012195/0.00629835}{3}.917085 \cdot 10^{-6}(days) \implies b' = 1.00188 \cdot 10^{-2}$, and hence:

$$\rho_1 = 1.00188 \cdot 10^{-2} + 7.584343 \cdot 10^{-6}(365) = 1.2787\%$$

We now have all the necessary information to induce the 1,3 and 5 year default rates implied by a Markov Chain Model: As we saw in section 6.1.1 and 6.1.4, we have:

$$\mathbb{P}(\text{Solvent for } t \in (1]) = q = e^{-h_1} = e^{-(\rho_1 - r_1)} = e^{-(1.2787\% - 0.6284\%)} = e^{-(.006503)} = 0.99352$$

And therefore:

$$\mathbb{P}(\text{Solvent for } t \in (0,k]) = \big(\mathbb{P}(\text{Solvent for } t \in (0,1])\big)^k = 0.99352^k$$

And hence:

$$\mathbb{P}(\text{Solvent for } t \in (0,3]) = 0.98069, \quad \text{and} \quad \mathbb{P}(\text{Solvent for } t \in (0,5]) = 0.96802$$

Which implies the following final answer:

$$\mathbb{P}(\text{Default for } t \in (0,1]) = 1 - q = 0.648\%$$
$$\mathbb{P}(\text{Default for } t \in (0,1]) = 1 - q^3 = 1.931\%$$
$$\mathbb{P}(\text{Default for } t \in (0,1]) = 1 - q^5 = 3.198\%$$

7 Merton Model

7.1 Necessary Theory

Let us begin by defining the following notation:

V_0 := Value of company's assets today.
V_T := Value of company's assets at time T.
E_0 := Value of company's equity today.
E_T := Value of company's equity at time T.
D := Amount of debt interest and principal due to be repaid at time T.
σ_V := Volatility of assets (assumed constant).
σ_E := Instantaneous volatility of equity - the standard deviation of the daily log returns (over the past year).

If $V_T < D$, it is (at least in theory) rational for the company to default on the debt at time T. The value of the equity is then zero. If $V_T > D$, the company should make the debt repayment at time

T and the value of the equity at this time is $V_T - D$. Merton's model, therefore, gives the value of the firm's equity at time T as

$$E_T = \max(V_T - D, 0)$$

This shows that the equity of a company is a call option on the value of the assets of the company with a strike price equal to the repayment required on the debt. The Black-Scholes-Merton formula gives the value of the equity today as:

$$E_0 = V_0\Phi(d_1) - De^{-rT}\Phi(d_2) \tag{14}$$

where:

$$\Phi(x) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}dt, \quad d_1 := \frac{\log(V_0/D) + (r + \sigma_V^2/2)T}{\sigma_V\sqrt{T}}, \quad d_2 := d_1 - \sigma_V\sqrt{T}$$

Under Merton's model, the company defaults when the option is not exercised. The probability of this can be shown to be $\Phi(-d_2)$. To calculate this, we require V_0 and σ_V. Neither of these are directly observable. However, if the company is publicly traded, we can observe E_0. This means that equation the way we defined E_0 above provides one condition that must be satisfied by V_0 and σ_V. We can also estimate σ_E. From a result in stochastic calculus known as Ito's lemma:

$$\sigma_E E_0 = \frac{\partial E}{\partial V}\sigma_V V_0$$

Where $\frac{\partial E}{\partial V}$ is the delta of the equity. It can be easily shown that:

$$\frac{\partial E}{\partial V} = \Phi(d_1) \implies \sigma_E E_0 = \Phi(d_1)\sigma_V V_0 \tag{15}$$

This provides another equation that must be satisfied by V_0 and σ_V. The Equations above thus provide a pair of simultaneous equations that can be solved for V_0 and σ_V. Therefore, after solving for V_0 and σ_V, we can apply these number to solve for the probability of default, defined as:

$$\mathbb{P}(\text{Default for } t \in (0,T]) = \frac{E_0 - V_0\Phi(d_1) + De^{-rT}}{De^{-rT}} = \Phi(-d_2) = 1 - \Phi(d_2)$$

7.2 The Theory Applied to the Question at Hand

Firstly, we consult Yahoo Finance to find out the following information about Bell Canada:

Variable	Value
# of Shares Outstanding	898.54M
Total Debt	21.359B
Stock Price on 2017-04-04	$59.67

Therefore, $E_0 :=$ Value of company's equity today $= (59.67\$)(898.54M) = 53.616B\$$.

We next download Bell's Daily Adjusted Close (adjusted to account for dividend payments) over the past year (I.e. from 2016-04-04). Let us call the adjusted Close time series y_t, $t \in \{1, \ldots, 253\}$. We then calculated σ_E as follows:

First define $r_t := \log(y_t) - \log(y_{t-1}), t \in \{2, \ldots, 253\}$. Then:

$$\sigma_E = \left(\frac{1}{252-1}\sum_{i=2}^{253}(r_i - \hat{r})\right)^{1/2}, \quad \hat{r} = \frac{1}{252}\sum_{j=2}^{253} r_j$$

This calculation yields $\sigma_E = 0.6233157\%$.

We also quickly note here that we define D as:

$$D := \text{Total Debt} \cdot e^{r_T \cdot T}$$

7.2.1 T=1

We use the RMFI Software v1.00 to carry out solving for the necessary nonlinear equations. Since these calculations are pretty simple, and since Bell's Delta will be around 1, we solve for σ_V and E_0 through some rudimentary trial and error; I.e., guess a beginning answer, then update with a better guess until we are within (say) 3-4 significant digits of satisfying each equation.

For $T = 1$, we make use of the additional data: $r = 0.6373\%$, and we (as an assumption) make use of Bell's bond which Matures on 2018-03-16 since this is only 19 days away from exactly one year from today. We recall that this bond has a price of 102.94, and coupon of 4.40. Furthermore, since $r = 0.6373\%$, we have $D = 21.59317B\$$.

We now make use of our software to help find σ_V and V_0 s.t. the following two equations are satisfied:

$$\sigma_E E_0 = 0.334196209 = \Phi(d_1)\sigma_V V_0 \tag{16}$$

and:

$$E_0 = 53.616 = V_0\Phi(d_1) - De^{-rT}\Phi(d_2) \tag{17}$$

where:

$$\Phi(x) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}dt, \quad d_1 := \frac{\log(V_0/1.044) + (r + \sigma_V^2/2)1}{\sigma_V\sqrt{1}}, \quad d_2 := d_1 - \sigma_V\sqrt{1}$$

We can see that $V_0 = 75.113$ and $\sigma_V = 0.445\%$ satisfy these equations. Therefore, by plugging in these values, we have $d_2 = 280.309$, and hence:

$$\mathbb{P}(\text{Default for } t \in (0, 1]) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-280.309} e^{-t^2/2}dt << 10^{-100}$$

7.2.2 T=3 & T=5

We carry out in the exact same methodology, as above. Solving the necessary equations but with $r = 0.849\%$, $T = 3$, and $D = 21.9100$, we get $\sigma_V = 0.4457\%$, $V_0 = 74.975$. This therefore implies:

$$\mathbb{P}(\text{Default for } t \in (0, 3]) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-160.452} e^{-t^2/2}dt << 10^{-100}$$

Finally, we carry out once more in the exact same methodology, as two above for $T = 5$. we have here $r = 1.089\%$, $T = 5$, $D = 22.554242$. We thus get from solving the same nonlinear equation (but adapted for the above changes): $\sigma_V = 0.4457$, $V_0 = 74.975$. This therefore implies $d_2 = 121.615$, and hence:

$$\mathbb{P}(\text{Default for } t \in (0, 5]) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-121.615} e^{-t^2/2}dt << 10^{-100}$$

8 Results Presented in Tables and Charts

We summarize our results in chart and table format below:

8.1 Tables

Under Markov Chain Model

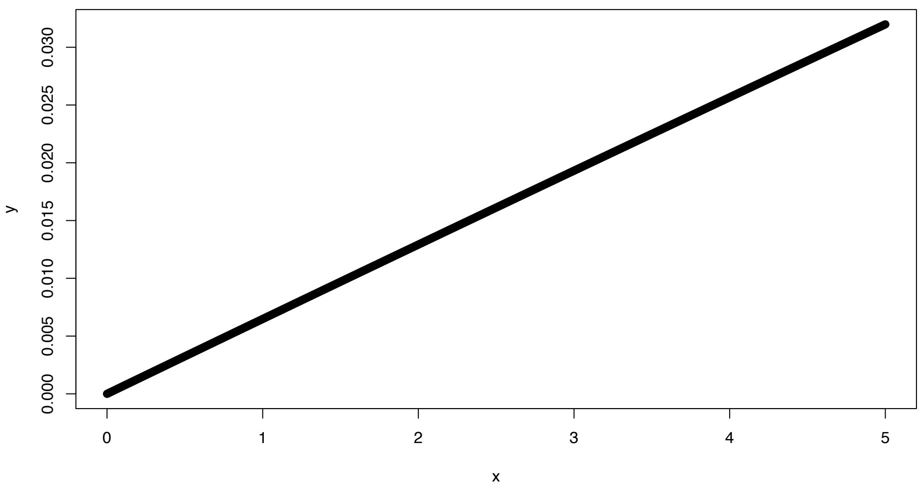
$$\mathbb{P}(\text{Default for } t \in (0,1]) = 1 - q = 0.648\%$$
$$\mathbb{P}(\text{Default for } t \in (0,1]) = 1 - q^3 = 1.931\%$$
$$\mathbb{P}(\text{Default for } t \in (0,1]) = 1 - q^5 = 3.198\%$$

Under Merton Model

$$\mathbb{P}(\text{Default for } t \in (0,1]) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-280.309} e^{-t^2/2} dt$$
$$\mathbb{P}(\text{Default for } t \in (0,3]) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-160.452} e^{-t^2/2} dt$$
$$\mathbb{P}(\text{Default for } t \in (0,5]) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-121.615} e^{-t^2/2} dt$$

8.2 Charts

Under Markov



And we do not present a plot for the Merton Model because our values are too close to zero to be of great visual significance.

9 References

References

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