MATH 2030 Elementary Probability/ Winter 2014

Test 1/ Feb. 3, 2014

Student Name: **ID-No.:**

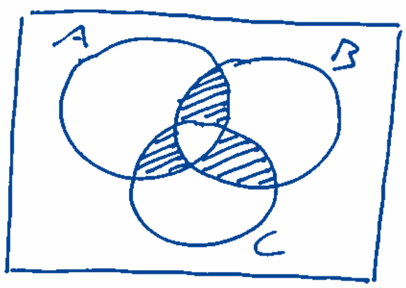
You have 50 minutes to solve the following problems: Show your complete work. Permitted aids: calculator and writing utensils; 1 written help-sheet of standard letter size with notes and formulas.

Problem 1 **(20 marks)**

Let A,B, and C be events which are mutually independent, with probabilities p_A, p_B, and p_C. Let N be the number of events which occur.

a) What is the event {N=2} in terms of A,B, and C?
b) What is the probability of this in terms of p_A, p_B, and p_C?

a)



the event $N=2$ in the Venn diagram

$$\{N=2\} = [(A\cap B)\cup(A\cap C)\cup(B\cap C)]\cap[A\cap B\cap C]$$

b) $P(N=2) = P(A\cap B) + P(A\cap C) + P(B\cap C) - 3P(A\cap B\cap C)$

$= P(A)\cdot P(B) + P(A)\cdot P(C) + P(B)\cdot P(C) - 3P(A)\cdot P(B)\cdot P(C)$

because of mutual independence

$= p_A\cdot p_B + p_A\cdot p_C + p_B\cdot p_C - 3p_A\cdot p_B\cdot p_C$.

Problem 2: **(20 marks)**

a) How many even two-digit numbers can be constructed out of the digits 3,4,5,6 and 7? Assume first that you may use the same digit again. Next answer this question assuming that you cannot use a digit more than once.

b) Six mountain climbers decide to divide into three groups for the final assault on the peak. The groups will be of size 1, 2, 3, respectively, and all manners of deployment (i.e. sequence of ascending) are considered. What is the total number of possible grouping and deploying?

2)a) i) $\square \square$

$\uparrow$ 10's digit $\uparrow$ 1's digit

To make the number even, one of the two digits 4,6 need to be selected for the 1's digit => 2 possible choices. If any digit can be chosen again, there are 5 choices for the 10's digit, to form a 2-digit number, i.e. $\underline{2 \times 5 = 10}$ possible ways to form a two-digit number. If each of the five digits 3,4,5,6 and 7 can be used only once, there are $\underline{2 \times 4 = 8}$ possible ways to form a two-digit number.

(ii)

b) Consider the different ways of grouping 6 climbers into team A (3 members), team B (2 members) and team C (1 member)

Suppose team C is formed first, then team B and then team A.

=> There are 6 ways of assigning a climber to team C; $\frac{5 \cdot 4}{2 \cdot 1}$ ways of choosing among the 5 remaining climbers 2 individuals to form team B; and the last 3 climbers form team A. => There are, in this order $6 \cdot \frac{5 \cdot 4}{2}$ different ways of grouping.

2)b) cont'd

Suppose that first team A is formed, then team C and then team B:
There are $\frac{6\cdot5\cdot4}{3\cdot2\cdot1}\cdot3\cdot\frac{2\cdot1}{2\cdot1}=\frac{6\cdot5\cdot4}{2}$

different ways of grouping. Likewise it is found that for any sequence of forming teams A, B and C there are $\frac{6\cdot5\cdot4}{2}$ different ways of grouping climbers into teams.

The teams, once formed, can be deployed (with ordering!) in $3\cdot2\cdot1$ different ways.

$\Rightarrow$ There are $\frac{6\cdot5\cdot4}{2}\cdot3\cdot2\cdot1=60\cdot6=360$

different ways of grouping and deploying.

Problem 3: **(20 marks)**

A certain firm has plants A,B, and C, producing respectively 35%, 15%, and 50%, of the total output. The probabilities of a nondefective product are, respectively, 0.75, 0.95, and 0.85. A customer receives a defective product. What is the probability that it came from plant C? (Hint: Apply Bayes' Rule.)

Denote by: A: A randomly selected item was produced in plant A;

B: ~ plant B;

C: ~ plant C.

$P(A) = 0.35$; $P(B) = 0.15$; $P(C) = 0.50$

Denote by: N: A randomly selected item/product is nondefective.

$P(N|A) = 0.75 \Rightarrow P(N^c|A) = 1-0.75 = 0.25$

$P(N|B) = 0.95 \Rightarrow P(N^c|B) = 1-0.95 = 0.05$

$P(N|C) = 0.85 \Rightarrow P(N^c|C) = 1-0.85 = 0.15$

P(product comes from plant C | product is defective) $= P(C|N^c)$

By Bayes' rule:

$$P(C|N^c) = \frac{P(C \cap N^c)}{P(N^c)} = \frac{P(N^c|C)\cdot P(C)}{P(N^c|A)\cdot P(A) + P(N^c|B)\cdot P(B) + P(N^c|C)\cdot P(C)}$$

$$= \frac{0.15 \cdot 0.50}{0.25\cdot 0.35 + 0.05\cdot 0.15 + 0.15\cdot 0.5}$$

$$= \underline{\underline{0.518}}$$

Problem 4: **(20 marks)**

Suppose that in 4-child families, each child is equally likely to be a boy or a girl, independently of the others. Which would then be more common, 4-child families with 2 boys and 2 girls, or 4-child families with different numbers of boys and girls? What would be the relative frequencies?

Denote by

X: number of boys in a 4-children family

$X \sim Bin(4, \frac{1}{2})$

$P(X=2) = \binom{4}{2} \cdot (\frac{1}{2})^2 \cdot (\frac{1}{2})^{4-2} = 6 (\frac{1}{2})^4 = 0.375$

probabilities for different numbers of boys and girls in a 4-children family:

$P(X=0) = \binom{4}{0} \cdot (\frac{1}{2})^0 \cdot (\frac{1}{2})^4 = \frac{1}{16}$

$P(X=1) = \binom{4}{1} \cdot (\frac{1}{2})^1 \cdot (\frac{1}{2})^3 = \frac{1}{4}$

$P(X=3) = \binom{4}{3} \cdot (\frac{1}{2})^3 \cdot (\frac{1}{2})^1 = \frac{1}{4}$

$P(X=4) = \binom{4}{4} \cdot (\frac{1}{2})^4 \cdot (\frac{1}{2})^0 = \frac{1}{16}$

$\Rightarrow P(X \neq 2) = \frac{10}{16} = 0.625$

$\Rightarrow$ families with different numbers of boys and girls are more common.

Problem 5: **(20 marks)**

A box contains 50 black balls and 30 red balls. Four balls are drawn at random from the box, one after the other, without replacement. Find the chance that:

a) All four balls are black;
b) Exactly three balls are black;
c) The first ~~read~~ ball appears on the last draw.

Let X = number of black balls among four drawn from the box without replacement.

a) $$P(X=4) = \frac{\binom{50}{4}\cdot\binom{30}{0}}{\binom{80}{4}}$$ (hypergeometric distribution)

$$= \underline{\underline{0.1456}}$$

b) $$P(X=3) = \frac{\binom{50}{3}\cdot\binom{30}{1}}{\binom{80}{4}}$$

$$= \frac{50\cdot 49\cdot 48}{3\cdot 2\cdot 1}\cdot\frac{30}{1}\cdot\frac{4\cdot 3\cdot 2\cdot 1}{80\cdot 79\cdot 78\cdot 77}$$

$$= \underline{\underline{0.3718}}$$

c) P(The first red ball appears on the last draw)

$$= \frac{P(X=3)}{4} = \frac{0.3718}{4} = \underline{\underline{0.093}}$$