McGill University

MATH-315

Winter — Practice for Final — 2017

Note: The text book has many examples! See what you need to practice here, then go back to the textbook. And, of course, if you need help, ask (email).

1. Solve the following ODEs. [*Hint: for the last part, try an idea from the method of undetermined coefficients.*]

   (i) $2y'' + y' - y = 5\sin(x)$

   (ii) $4y'' + 12y' + 9y = -2e^{-3x/2}$

   (iii) $y'' + y' + 4y = (x+1)^2$

   (iv) $x^2y'' + xy' - y = x^{101}$

2. Solve the following ODEs.

   (i) $y'' + 6y' + 9y = e^{-3x}\ln(x)$

   (ii) $y'' + 49y = \csc(7x)$

   (iii) $x^2y'' - 6xy' + 12y = \cos\left(\frac{1}{x}\right)$

3. Construct a general solution in terms of power series solutions for each of the following ODEs around the specified point. Specify y_1 and y_2, where the form of your solution is $y(x) = c_1y_1 + c_2y_2$, and determine an interval of convergence for each.

   (i) $y'' + xy' + y = 0\,, \quad x = 0$

   (ii) $(1-x^2)y'' + 2y = 0\,, \quad x = 0$

4. Find and classify all singular points of the following ODEs.

   (i) $x^2(1-x)^2y'' + 2xy' + 4y = 0$

   (ii) $x^2(1-x^2)y'' + \frac{2}{x}y' + 4y = 0$

5. Show that $x = 0$ is a regular singular point for the ODE

$$2x^2y'' + 3xy' + (x-1)y = 0\,.$$

   (i) Obtain the indicial equation and find its two roots r_1 and r_2.

   (ii) Given that these roots do not differ by an integer value both power series solutions about $x = 0$ have the form

$$y_k(x) = |x|^{r_k}\sum_{n=0}^{\infty} a_n x^n$$

   for $k = 1$ or $k = 2$ where r_k is either of the two roots of the indicial equation found above. Find the recursive relationship for the coefficients of each series as well as the first four terms of each series. (There is no need to determine the interval of convergence for these series.)

6. Obtain particular solutions for the following ODEs using the Laplace transform (use the table provided).

   (i) $y'' - y' + y = \delta(t-2)\,,\quad y(0) = 1,\ y'(0) = 0$

   (ii) $y'' - 5y = f(t)\,,\quad y(0) = 0,\ y'(0) = 2$ where $f(t)$ is defined as

$$f(t) = \begin{cases} 0\,, & \text{if } t < 4\,, \\ 7\,, & \text{if } t \geq 4\,. \end{cases}$$

7. Solve the following 2-by-2 systems of ODEs.

   (i) $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

   (ii) $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

   (iii) $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

8. Given the following initial conditions, determine particular solutions for the systems in the previous question.

   (i) $x(0) = 1\,,\quad y(0) = 1$

   (ii) $x(0) = 3\,,\quad y(0) = -2$

   (iii) $x(0) = \sqrt{2}\,,\quad y(0) = -\sqrt{2}$

Answers

1. In the following A and B are arbitrary constants.

   (i) $y(x) = Ae^{x/2} + Be^{-x} - \frac{1}{2}(\cos(x) + 3\sin(x))$

   (ii) $y(x) = (A + Bx)e^{-3x/2} - \frac{1}{4}x^2 e^{-3x/2}$

   (iii) $y(x) = e^{-x/2}\left(A\cos\left(\frac{\sqrt{15}}{2}x\right) + B\sin\left(\frac{\sqrt{15}}{2}x\right)\right) + \frac{1}{32}(1 + 12x + 8x^2)$

   (iv) $y(x) = \frac{A}{x} + Bx + \frac{1}{10200}x^{101}$

2. In the following C and D are arbitrary constants.

   (i) $y(x) = (C + Dx)e^{-3x} + \frac{1}{4}e^{-3x}x^2(-3 + 2\ln(x))$

   (ii) $y(x) = C\cos(7x) + D\sin(7x) - \frac{1}{7}x\cos(7x) + \frac{1}{49}\sin(7x)\ln(|\sin(7x)|)$

   (iii) $y(x) = Cx^3 + Dx^4 + (6x^4 - x^2)\cos\left(\frac{1}{x}\right) + 4x^3\sin\left(\frac{1}{x}\right)$

3. In each of the following the general solution is given by $y(x) = c_1y_1(x) + c_2y_2(x)$ where c_1 and c_2 are arbitrary constants. The interval of absolute convergence (IoC) for each series follows.

   (i) $y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!2^n}x^{2n}$, $\quad y_2(x) = \sum_{n=0}^{\infty} \frac{(-1)^n n!2^n}{(2n+1)!}x^{2n+1}$ IoC: $(-\infty, +\infty)$.

   (ii) $y_1(x) = 1 - x^2$ (for all $x \in \mathbb{R}$) , $y_2(x) = \sum_{n=0}^{\infty} \frac{1}{1 - 4n^2}x^{2n+1}$ IoC: $(-1, +1)$.

4. Points are either regular singular points or irregular singular points.

   (i) $x = 0$ (regular) and $x = 1$ (irregular).

   (ii) $x = -1$ (regular), $x = 0$ (irregular) and $x = 1$ (regular).

5. In classifying the singular point we obtain p_0 and q_0 as the values of the limits of $p(x) = xb(x)/a(x)$ and $q(x) = x^2c(x)/a(x)$, respectively, as $x \to 0$.

   (i) Here $p_0 = 3/2$ and $q_0 = -1/2$ giving the indicial equation

   $$r(r-1) + \frac{3}{2}r - \frac{1}{2} = r^2 + \frac{1}{2}r - \frac{1}{2} = 0\,.$$

   The roots of this equation are $r_1 = -1$ and $r_2 = 1/2$.

   (ii) For $r_1 = -1$ we have the following recursive relationship for $n \geq 1$:

   $$a_n = -\frac{1}{n(2n-3)}a_{n-1}\,.$$

   The corresponding series starts off as follows

   $$y_1(x) = a_0\left(\frac{1}{x} + 1 - \frac{1}{2}x + \frac{1}{18}x^2 - \frac{1}{360}x^3 + \dots\right)\,.$$

   Similarly for $r_2 = 1/2$ we have the following recursive relationship for $n \geq 1$:

   $$a_n = -\frac{1}{n(2n+3)}a_{n-1}\,.$$

   The corresponding series starts off as follows

   $$y_2(x) = a_0\left(x^{1/2} - \frac{1}{5}x^{3/2} + \frac{1}{70}x^{5/2} - \frac{1}{1890}x^{7/2} - \frac{1}{83160}x^{9/2} + \dots\right)\,.$$

6. Corresponding to the solutions from the previous question.

   (i) $y(t) = e^{t/2}\left(\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)\right) + u_2(t)\,\frac{2}{\sqrt{3}}e^{(t-2)/2}\sin\left(\frac{\sqrt{3}}{2}(t-2)\right)$

   (ii) $y(t) = \frac{1}{\sqrt{5}}\left(e^{+\sqrt{5}t} - e^{-\sqrt{5}t}\right) + \frac{7}{10}u_4(t)\left(-2 + e^{+\sqrt{5}(t-4)} + e^{-\sqrt{5}(t-4)}\right)$

7. In the following A and B are arbitrary constants.

   (i) $\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} 1 \\ -1 \end{pmatrix} + B\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{2t}$

   (ii) $\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} \cos(5t) \\ -\sin(5t) \end{pmatrix}e^{3t} + B\begin{pmatrix} \sin(5t) \\ \cos(5t) \end{pmatrix}e^{3t}$

   (iii) $\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{t} + B\left[\begin{pmatrix} -1 \\ 0 \end{pmatrix}e^{t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}te^{t}\right]$

8. Particular solutions which correspond to the general solutions found in the answer above.

(i) $$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$$

(ii) $$\begin{pmatrix} x \\ y \end{pmatrix} = 3 \begin{pmatrix} \cos(5t) \\ -\sin(5t) \end{pmatrix} e^{3t} - 2 \begin{pmatrix} \sin(5t) \\ \cos(5t) \end{pmatrix} e^{3t}$$

(iii) $$\begin{pmatrix} x \\ y \end{pmatrix} = -\sqrt{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{t} - \sqrt{2} \left[\begin{pmatrix} -1 \\ 0 \end{pmatrix} e^{t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} te^{t} \right]$$