MAT224H1F: Linear Algebra II
Midterm #2 Review

General Questions

The following are some very general questions you should be able to answer. If you have difficulty answering a particular question, review the relevant parts of the lecture notes and problem sets.

- How can we determine whether a set of vectors is linearly independent, or whether it is linearly dependent?
- How can we determine the dimension of a vector space or subspace?
- Let V be an n-dimensional vector space and let $\beta = \{v_1, v_2, \ldots, v_m\}$. If $m < n$, is it possible for β to be linearly independent? Is it possible for β to span V?
- Let V be an n-dimensional vector space and let $\beta = \{v_1, v_2, \ldots, v_m\}$. If $m > n$, is it possible for β to be linearly independent? Is it possible for β to span V?
- Let V be an n-dimensional vector space and let $\beta = \{v_1, v_2, \ldots, v_m\}$. If $m = n$, is it possible for β to be linearly independent? Is it possible for β to span V? Is it easier than usual to determine whether β is a basis of V?
- Let V be a vector space and let $\beta = \{v_1, v_2, \ldots, v_n\}$ be a basis of V. Given any $v \in V$, must it be possible to write v as a linear combination of the vectors in β? If so, might there be another way to express v as a linear combination of these vectors?
- How might we find a basis of a subspace U (ex. $U = \{p(x) \in P_3(\mathbb{R}) : p'(4) - p(0) = 0\}$)?
- If we are given a vector, how can we find its coordinates with respect to a basis? If we are given the coordinates of a vector with respect to a basis, how can we find the original vector?
- What does it mean for $T : V \to W$ to be the zero transformation?
- What does it mean for $T : V \to V$ to be the identity transformation?
- If $T : V \to W$ is a linear transformation, α is a basis of V, and β is a basis of W, what is the procedure for finding $[T]_\alpha^\beta$?
- How can we find a basis for the kernel of a linear transformation?
- How can we find a basis for the image of a linear transformation?
- What are some ways to determine whether a linear transformation is injective /surjective?
- Let $T : V \to W$ be a linear transformation. If $\dim(V) < \dim(W)$, is it possible for T to be injective? Is it possible for T to be surjective?
- Let $T : V \to W$ be a linear transformation. If $\dim(V) > \dim(W)$, is it possible for T to be injective? Is it possible for T to be surjective?
- What are some ways to determine whether a linear transformation is an isomorphism?
- How might we find a formula for the inverse of an isomorphism?

Practice Problems

The following are some practice problems to aid the review process. While solving these problems will be useful to you, these problems are **absolutely not** intended to replace the lecture notes and problem sets as resources. Please solve these problems **in addition to** reviewing the lecture notes and problem sets.

1. Consider the bases $\alpha = \{1, x, x^2\}$ of $P_2(\mathbb{R})$ and $\beta = \{1, x, x^2, x^3\}$ of $P_3(\mathbb{R})$. Also, consider the linear transformation $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ defined by $T(p(x)) = x^2 p'(x)$.

   (a) Is T an isomorphism? Justify your answer.

   (b) Find $[T]_\alpha^\beta$.

   (c) Find a basis of $\mathrm{Im}(T)$.

   (d) Is T injective? Justify your answer.

2. Consider the bases $\alpha = \{1, x, x^2, x^3\}$ of $P_3(\mathbb{R})$ and $\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ of $M_{2\times 2}(\mathbb{R})$. Also, consider the linear transformation $T : P_3(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by

$$T(p(x)) = \begin{bmatrix} p(0) & p'(0) \\ p''(0) & p'''(0) \end{bmatrix},$$

   where $p'(x)$, $p''(x)$, and $p'''(x)$ are the first, second, and third derivatives of $p(x)$, respectively.

   (a) Find $[T]_\alpha^\beta$.

   (b) Is T an isomorphism? Justify your answer.

   (c) If T is an isomorphism, find a formula for $T^{-1}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$.

3. Consider the basis $\alpha = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ of $M_{2\times 2}(\mathbb{R})$, and let $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be the linear transformation defined by

$$T(X) = X^T + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} X.$$

   (a) Find $[T]_\alpha^\alpha$.
   (b) Show that T is an isomorphism.
   (c) If $A \in M_{2\times 2}(\mathbb{R})$, show that there exists a unique $X \in M_{2\times 2}(\mathbb{R})$ such that

$$A = X^T + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} X.$$

4. Let V be an m-dimensional vector space, W an n-dimensional vector space, and $T : V \to W$ a linear transformation.

   In each of the following cases, determine whether the given statement is true or false. Justify each of your conclusions.

   (a) If $m \geq n$, then T is surjective.

   (b) If $m \geq n$ and T is injective, then T is surjective.

   (c) If $m \leq n$, then T is injective.

   (d) If $m \leq n$ and T is surjective, then T is an isomorphism.

   (e) If $m = n$, then T is an isomorphism.

5. A square matrix $N \in M_{n\times n}(\mathbb{R})$ is called *nilpotent* if $N^k = 0$ for some positive integer k.

   Let $N \in M_{n\times n}(\mathbb{R})$ be a nilpotent matrix and consider the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T(v) = Nv$.

   (a) Show that T is not injective.

   Hint: Using the fact that N is nilpotent, try to find examples of vectors in $\ker(T)$.

   (b) Is T surjective? Justify your answer.

6. Let V be an n-dimensional vector space with $n \geq 2$, and let $\alpha = \{v_1, v_2, \ldots, v_n\}$ be a basis of V. Also, let $T : V \to V$ be a linear transformation with the following two properties.

   – $T(v_1) = v_1$
   – For all $i \geq 2$, $T(v_i) \in \text{span}\{v_1, v_2, \ldots, v_{i-1}\}$, meaning that $T(v_2) \in \text{span}\{v_1\}$, $T(v_3) \in \text{span}\{v_1, v_2\}$, $T(v_4) \in \text{span}\{v_1, v_2, v_3\}$, etc.

   Show that T is not an isomorphism.

   Hint: Consider the form of $[T]_\alpha^\alpha$.