MAT224H1F: Linear Algebra II
Midterm #1 Review

General Questions

The following are some very general questions you should be able to answer. If you have difficulty answering a particular question, review the relevant parts of the lecture notes and problem sets.

- What must be done to show something is a subspace?
- How can we determine whether a given vector v is in $\text{span}\{v_1, v_2, \ldots, v_n\}$?
- How can we determine whether a set of vectors is linearly independent, or whether it is linearly dependent?
- How can we determine the dimension of a vector space or subspace?
- Let V be an n-dimensional vector space and let $\beta = \{v_1, v_2, \ldots, v_m\}$. If $m < n$, is it possible for β to be linearly independent? Is it possible for β to span V?
- Let V be an n-dimensional vector space and let $\beta = \{v_1, v_2, \ldots, v_m\}$. If $m > n$, is it possible for β to be linearly independent? Is it possible for β to span V?
- Let V be an n-dimensional vector space and let $\beta = \{v_1, v_2, \ldots, v_m\}$. If $m = n$, is it possible for β to be linearly independent? Is it possible for β to span V? Is it easier than usual to determine whether β is a basis of V?
- Let V be a vector space and let $\beta = \{v_1, v_2, \ldots, v_n\}$ be a basis of V. Given any $v \in V$, must it be possible to write v as a linear combination of the vectors in β? If so, might there be another way to express v as a linear combination of these vectors?
- How might we find a basis of a subspace U
  (ex. $U = \{p(x) \in P_3(\mathbb{R}) : p'(4) - p(0) = 0\}$)?

Practice Problems

The following are some practice problems to aid the review process. While solving these problems will be useful to you, these problems are **absolutely not** intended to replace the lecture notes and problem sets as resources. Please solve these problems **in addition to** reviewing the lecture notes and problem sets.

1. Consider the vector space $V = \mathbb{R}^5$ and let $U = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \in \mathbb{R}^5 : a_1 - 2a_3 = 0 \right\}$.

   (a) Show that U is a subspace of $\mathbb{R}^5$.

   (b) Find a basis of U.

   (c) Find $\dim(U)$.

2. Consider the vector space $V = M_{2\times 2}(\mathbb{R})$ and let

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}.$$

(a) Is S linearly independent, or is it linearly dependent? Justify your answer.

(b) Is S a basis of $M_{2\times 2}(\mathbb{R})$? Justify your answer.

(c) Express $\begin{bmatrix} -2 & 0 \\ 2 & 5 \end{bmatrix}$ as a linear combination of the vectors in S.

3. Consider the vector space $V = M_{2\times 2}(\mathbb{R})$ and let

$$S = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$$

(a) Is S a basis of $M_{2\times 2}(\mathbb{R})$? Justify your answer.

(b) Does S span $M_{2\times 2}(\mathbb{R})$? Justify your answer.

4. Consider the vector space $V = P_4(\mathbb{R})$ and let

$$S = \{1 - 8x + x^3 - 3x^4,\ x + 6x^2 - 2x^3 + x^4,\ x^2 + 10x^3 - x^4,\ x^3 - x^4\}.$$

(a) Is S linearly independent, or is it linearly dependent? Justify your answer.

(b) Does S span $P_4(\mathbb{R})$? Justify your answer.

5. Let V be a vector space and suppose that $v_1, v_2, \ldots, v_n \in V$.

(a) If $\{v_1, v_2, \ldots, v_n\}$ is linearly independent, prove that $\{v_1, v_2 + v_1, v_3 + v_1, \ldots, v_n + v_1\}$ is also linearly independent.

(b) If $\{v_1, v_2, \ldots, v_n\}$ is linearly dependent, must it be true that $v_1 \in \operatorname{span}\{v_2, \ldots, v_n\}$? Either prove this must be true, or give a specific counter-example.

6. Let V be a vector space and assume that $\{v_1, v_2\}$ is a basis of V. Consider the set $S = \{cv_1, v_1 + v_2\}$, where $c \in \mathbb{R}$.

(a) Give an example of a value of c for which S is linearly dependent.

(b) Find all values of c for which S is linearly dependent.

(c) Find all values of c for which S is linearly independent.

(d) Find all values of c for which S is a basis of V.

(e) Find all values of c for which S spans V.