MAT224H1F: Linear Algebra II
Final Exam Review Sheet

General Comments

This document is designed to help students prepare for the final exam in MAT224F. However, it is **absolutely not** intended to summarize or be a substitute for the lecture notes and problem sets.

General Questions

The following are some very general questions you should be able to answer.

- What must be done to show something is a subspace?
- How can we determine whether a given vector v is in $\text{span}\{v_1, v_2, \ldots, v_n\}$?
- How can we determine whether a set of vectors is linearly independent, or whether it is linearly dependent?
- How can we determine the dimension of a vector space or subspace?
- Let V be an n-dimensional vector space and let $\beta = \{v_1, v_2, \ldots, v_m\}$. If $m < n$, is it possible for β to be linearly independent? Is it possible for β to span V?
- Let V be an n-dimensional vector space and let $\beta = \{v_1, v_2, \ldots, v_m\}$. If $m > n$, is it possible for β to be linearly independent? Is it possible for β to span V?
- Let V be an n-dimensional vector space and let $\beta = \{v_1, v_2, \ldots, v_m\}$. If $m = n$, is it possible for β to be linearly independent? Is it possible for β to span V? Is it easier than usual to determine whether β is a basis of V?
- How might we find a basis of a subspace U (ex. $U = \{p(x) \in P_3(\mathbb{R}) : p'(4) - p(0) = 0\}$)?
- If we are given a vector, how can we find its coordinates with respect to a basis? If we are given the coordinates of a vector with respect to a basis, how can we find the original vector?
- What does it mean for $T : V \to W$ to be the zero transformation?
- What does it mean for $T : V \to V$ to be the identity transformation?
- If $T : V \to W$ is a linear transformation, α is a basis of V, and β is a basis of W, what is the procedure for finding $[T]_\alpha^\beta$?
- How can we find a basis for the kernel of a linear transformation?
- How can we find a basis for the image of a linear transformation?
- What are some ways to determine whether a linear transformation is injective /surjective?
- Let $T : V \to W$ be a linear transformation. If $\dim(V) < \dim(W)$, is it possible for T to be injective? Is it possible for T to be surjective?
- Let $T : V \to W$ be a linear transformation. If $\dim(V) > \dim(W)$, is it possible for T to be injective? Is it possible for T to be surjective?
- What are some ways to determine whether a linear transformation is an isomorphism?
- How can we find a formula for the inverse of an isomorphism?
- How can we find the characteristic polynomial and eigenvalues of a linear transformation?

- Are two eigenvectors always linearly independent?
- What is the procedure for finding a basis of an eigenspace?
- What are some ways to determine whether a linear transformation is diagonalizable?
- What are some examples of non-diagonalizable linear transformations?
- If we have found a basis of a subspace, how can we find an orthogonal basis?
- How can we determine whether a linear transformation is symmetric?
- What is the Spectral Theorem?
- When are two eigenvectors orthogonal?

PRACTICE PROBLEMS

1. Let $U = \{p(x) \in P_3(\mathbb{R}) : p(1) + p'(0) = 0\}$ and consider the inner product on $P_3(\mathbb{R})$ defined by
$$\langle p(x), q(x) \rangle = \int_0^1 p(x)q(x)dx.$$
   (a) Show that U is a subspace of $P_3(\mathbb{R})$.

   (b) Find a basis of U.

   (c) Find an orthogonal basis of U.

2. Consider the vector space $V = M_{2\times2}(\mathbb{R})$, and let
$$S = \left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}.$$
   (a) Is S a basis of $M_{2\times2}(\mathbb{R})$? Justify your answer.

   (a) Is S linearly independent, or is it linearly dependent? Justify your answer.

   (b) Does S span $M_{2\times2}(\mathbb{R})$? Justify your answer.

3. Consider the linear transformation $T : M_{2\times2}(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ defined by $T(X) = 3X^T + 3X$.

   (a) Find a basis of $\text{Im}(T)$.

   (b) Using your answer to part (a), determine $\dim(\ker(T))$.

   (c) Is T injective? Justify your answer.

   (d) Is T surjective? Justify your answer.

   (e) Is T an isomorphism? Justify your answer.

4. Consider the linear transformation $T : P_3(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ defined by
$$T(p(x)) = \begin{bmatrix} p(0) & p(1) \\ p(-2) & p(4) \end{bmatrix}.$$

(a) Find $[T]_\alpha^\beta$, where $\alpha = \{1, x, x^2, x^3\}$ and $\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ are the standard bases of $P_3(\mathbb{R})$ and $M_{2\times 2}(\mathbb{R})$, respectively.

(b) Show that $\alpha' = \{1, x-2, x^2-x+3, x^3-2x^2-x+4\}$ is a basis of $P_3(\mathbb{R})$ and find the change of basis matrices $[I]_{\alpha'}^{\alpha}$ and $[I]_{\alpha}^{\alpha'}$.

(c) Show that $\beta' = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis of $M_{2\times 2}(\mathbb{R})$ and find the change of basis matrices $[I]_{\beta'}^{\beta}$ and $[I]_{\beta}^{\beta'}$.

(d) Using your answers to parts (b) and (c), find $[T]_{\alpha'}^{\beta'}$.

5. Consider the linear transformation $T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ defined by $T(p(x)) = p(x) + (2x+1)p'(x) - xp''(x)$.

   (a) Find $[T]_\alpha^\alpha$, where $\alpha = \{1, x, x^2, x^3\}$ is the standard basis of $P_3(\mathbb{R})$.

   (b) Is T an isomorphism? If so, find a formula for $T^{-1}(a_0 + a_1 x + a_2 x^2 + a_3 x^3)$.

   (c) Is T diagonalizable? If so, find a basis β of $P_3(\mathbb{R})$ such that $[T]_\beta^\beta$ is diagonal.

6. Consider the inner product on $M_{n\times n}(\mathbb{R})$ defined by $\langle X, Y\rangle = \mathrm{Tr}(X^T Y)$. Also, consider the linear transformation $T : M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$ given by $T(X) = X - X^T$.

   (a) Show that T is symmetric.

   (b) Determine whether T is diagonalizable.

7. Consider the inner product on $M_{2\times 2}(\mathbb{R})$ defined by $\langle X, Y\rangle = \mathrm{Tr}(X^T Y)$. Also, consider the linear transformation $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ given by $T(X) = X^T - X$

   (a) Find a basis of each eigenspace of T.

   (b) One of the eigenspaces is 3-dimensional while the other is 1-dimensional. Verify that the basis vector from the 1-dimensional eigenspace is orthogonal to each basis vector from the 3-dimensional eigenspace. Is this consistent with the Spectral Theorem?

8. Consider the linear transformation $T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ defined by

$$T(a_0 + a_1 x + a_2 x^2 + a_3 x^3) = a_1 + (a_2 + a_3)x^2 + a_3 x^3.$$

   (a) Is T diagonalizable? Justify your answer.

   (b) Let $\langle \cdot, \cdot \rangle$ be an inner product on $P_3(\mathbb{R})$. Is it possible for T to be symmetric? Justify your answer.

9. Let V be a finite-dimensional vector space and $T : V \to V$ a diagonalizable linear transformation with characteristic polynomial $p_T(\lambda)$. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the

eigenvalues of T, and let $d_1 = \dim(E_{\lambda_1}), d_2 = \dim(E_{\lambda_2}), \ldots, d_k = \dim(E_{\lambda_k})$. Show that

$$p_T(0) = \lambda_1^{d_1}\lambda_2^{d_2}\cdots\lambda_k^{d_k}.$$

Hint: Since T is diagonalizable, there exists a basis of V consisting of eigenvectors of T. Use this basis to help calculate $p_T(\lambda)$.

10. Let V be an inner product space[1] with orthogonal basis $\{v_1, v_2, \ldots, v_n\}$. If $v \in V$, show that

$$v = \frac{\langle v, v_1\rangle}{\langle v_1, v_1\rangle}v_1 + \frac{\langle v, v_2\rangle}{\langle v_2, v_2\rangle}v_2 + \ldots + \frac{\langle v, v_n\rangle}{\langle v_n, v_n\rangle}v_n.$$

Hint: Since $\{v_1, v_2, \ldots, v_n\}$ is a basis of V, one can write $v = a_1v_1 + a_2v_2 + \ldots + a_nv_n$ for some $a_1, a_2, \ldots, a_n \in \mathbb{R}$. Try to calculate $a_1, a_2, \ldots, a_n$ using the fact that $\{v_1, v_2, \ldots, v_n\}$ is an orthogonal basis.

11. Let V be an inner product space with basis $\{v_1, v_2, \ldots, v_n\}$ (not necessarily an orthogonal basis). If $v, w \in V$ and $\langle v, v_i\rangle = \langle w, v_i\rangle$ for all $i = 1, 2, \ldots, n$, show that $v = w$.

12. Let V be a finite-dimensional inner product space and $T : V \to V$ a linear transformation. Suppose that $\|T(v)\| = \|v\|$ for every vector $v \in V$.
    (a) Show that T is an isomorphism.
    (b) If $\lambda \in \mathbb{R}$ is an eigenvalue of T, show that $\lambda = 1$ or $\lambda = -1$.
    (c) Suppose also that T is diagonalizable and that -1 is not an eigenvalue of T. Show that T is the identity transformation on V.

13. Let V be a finite-dimensional inner product space, $T : V \to V$ a linear transformation, and $c \in \mathbb{R}$ a real number. Also, let $S : V \to V$ be the linear transformation defined by $S(v) = cT(v)$.
    (a) If T is injective, find all values of c for which S is surjective.
    (b) If T is surjective, find all values of c for which S is not an isomorphism.
    (c) If T is diagonalizable, find all values of c for which S is diagonalizable.
    (d) If T is diagonalizable, find all values of c for which TST is diagonalizable. [Note that TST and $T \circ S \circ T$ have the same meaning.]
    (e) If T is symmetric, find all values of c for which S is symmetric.
    (f) If T is symmetric, find all values of c for which S is diagonalizable.

[1] An inner product space is a vector space on which we have chosen a specific inner product.