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The University of British Columbia

Sessional Exams – 2015 Term 2
Mathematics 318 Probability with Physical Applications
Dr. G. Slade

Last Name: ____________________ **First Name:** ____________________

Student Number: ____

This exam consists of **8** questions worth **10** marks each.
No aids are permitted. There are tables on the last page.
Please show all work and calculations. Numerical answers need not be simplified.

Problem	total possible	score
1.	10	
2.	10	
3.	10	
4.	10	
5.	10	
6.	10	
7.	10	
8.	10	
total	80	

1. Consider 8-digit numbers where each digit is one of the ten integers 0,1,2,3,4,5,6,7,8,9.

(2 marks) (a) How many 8-digit numbers have 0 appear as a digit exactly 3 times?

(2 marks) (b) I choose 8-digit numbers at random until I obtain one that does not contain 0 as a digit. Let X be the number of attempts this takes. What is the distribution of X (give its name and any parameter(s)) and what is the probability that $X = 3$?

(2 marks) (c) I choose 25 8-digit numbers at random. Let N be the number of those 8-digit numbers that do not contain 0 as a digit. What is the distribution of N (give its name and any parameter(s)) and what is the probability that $N = 3$?

(2 marks) (d) What is the expected number of times that 0 appears as a digit in a randomly chosen 8-digit number?

(2 marks) (e) How many 8-digit numbers are there that have no two consecutive digits equal?

(10 marks) 2. Ms. Lee has just had a biopsy on a possibly cancerous tumour, and the result will be known tomorrow. Due to a family reunion about to take place, she does not want to hear any bad news in the next couple of days. If she tells the doctor to call only if the news is good, then if the doctor doesn't call she will know that the news is bad. So she asks the doctor to flip a (fair) coin when he learns the result. If it comes up heads, the doctor is to call if the news is good and not call if the news is bad. If it comes up tails, the doctor is not to call. Let p be the probability, before the biopsy is performed, that the tumour is cancerous. Determine the conditional probability that the tumour is cancerous given that the doctor does not call.

3. Let $X, X_1, X_2, X_3, \ldots$ be independent exponential random variables with parameter $\lambda = \frac{1}{2}$.

(1 marks) (a) Find $P(2 < X < 3)$.

(3 marks) (b) Determine the probability density function of X^2.

(3 marks) (c) Let $S_n = \sum_{i=1}^{n} X_i$. Find (approximately) $P(S_{49} > 100)$.

(3 marks) (d) Let U be a uniform random variable on $[0, 1]$, and suppose that X and U are independent. Find $P(X > U)$.

4. The parts in this question are not related to each other.

(3 marks) (a) Suppose that the events E, F are independent. Prove that the events E, F^c are also independent.

(4 marks) (b) Let U_1, U_2, U_3 be independent uniform random variables on $[0, 1]$. Let $M = \max\{U_1, U_2, U_3\}$. Determine the probability density function of the random variable M.

(3 marks) (c) Consider the simple symmetric 1-dimensional random walk started at 0, which takes steps $+1$ or -1 with equal probabilities $\frac{1}{2}$. If $S_0, S_1, \ldots, S_n$ are the positions of the random walk at times $0, 1, \ldots, n$, then the *range* R_n of the random walk is the number of distinct points the walk visits by time n. That is, $R_n = (\max_{i=0,\ldots,n} S_i) - (\min_{i=0,\ldots,n} S_i) + 1$. It is known (you need not prove this) that

$$ER_n \sim \left(\frac{8n}{\pi}\right)^{1/2} \quad \text{as } n \to \infty.$$

An Octave script is written to obtain an approximation to ER_n by simulating 10^4 n-step simple random walks started at 0 and taking the average of all of their ranges. The script is run for $n = 10, 20, 40, 80, 160, 320, 640, 1280, 2560, 5120, 10240$, and then $\log ER_n$ is plotted against $\log n$ for these n. If your plot is consistent with the theoretical result, then sketch a carefully labelled graph showing the output you would expect to obtain.

5. Customers arrive at a server according to a Poisson process of rate 12 per hour. Service is instantaneous, and the customers depart immediately after arrival. The process begins at time zero.

(2 marks) (a) What is the distribution of the number of customers that arrive during the first 10 minutes (give its name and any parameter(s))?

(2 marks) (b) What is the probability that exactly one customer arrives during the first 10 minutes?

(2 marks) (c) No customer arrives in the first 5 minutes. What is the probability that the server waits at least 10 additional minutes for a customer?

(2 marks) (d) Two customers arrive in the first 5 minutes. What is the expected number of customers during the first 10 minutes?

(2 marks) (e) Some customers are sent to another server for further processing; on average these customers comprise one percent of all customers. Of the first 40 customers, what is the probability that exactly two were sent on for further processing and 38 were not?

6. For this question, you may use the formula:

$$\sum_{x=1}^{\infty} \frac{1}{x^2} \cos(kx) = \frac{\pi^2}{6} - \frac{\pi|k|}{2} + \frac{k^2}{4} \quad \text{for } -\pi \le k \le \pi.$$

Let $X_1, X_2, \ldots$ be independent and identically distributed integer-valued random variables, with probability mass function, for $x \in \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$,

$$P(X_i = x) = \begin{cases} \frac{3}{\pi^2}\frac{1}{x^2} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Consider the random walk on $\mathbb{Z}$ whose position at time n is given by $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$ (for $n \ge 1$).

(1 marks) (a) Determine all communicating classes of this Markov chain.

(1 marks) (b) Let $x \in \mathbb{Z}$ be a state. Determine the period of x.

(2 marks) (c) What precisely does it mean to say that state 0 is recurrent or transient?

(3 marks) (d) Compute the characteristic function $\phi_1(k)$ of X_1.

(3 marks) (e) Is state 0 recurrent or transient? You may use fact that the expected number of visits to 0 is given by the integral

$$\int_{-\pi}^{\pi} \frac{1}{1-\phi_1(k)} \frac{dk}{2\pi}.$$

7. Smith gambles constantly at four different casinos, and when he leaves one casino he goes to another which is chosen from the other three with equal probabilities $\frac{1}{3}$. He owns just one umbrella, which he can store at the casinos. Each trip between casinos, he takes his umbrella if it is raining and if he has it at his starting casino, and he never takes the umbrella if it is not raining. It rains independently each trip with probability p.

(3 marks) (a) Let X_n be the number of umbrellas at his current location at the start of the n^{th} trip. This defines a Markov Chain. Write down its transition matrix.

(2 marks) (b) Suppose Smith has the umbrella at time n. What is the probability that he will also have it at time $n+2$?

(3 marks) (c) Determine the stationary distribution of the Markov Chain.

(2 marks) (d) In the long run, what fraction of trips does Smith get wet?

8. A total of $2m$ balls, m of which are white and m of which are black, are distributed between two urns in such a way that each urn contains m balls.

(2 marks) (a) Suppose we select m of the $2m$ balls uniformly at random, place them in urn number 1, and place the other m balls in urn number 2. What is the probability that the number of white balls in urn number 1 is equal to i, for $i = 0, 1, \ldots, m$?

(3 marks) (b) We say that the system is in state i ($i = 0, 1, \ldots, m$) if urn number 1 contains i white balls. We define a Markov chain on this state space as follows. At each step, we draw one ball from each urn, place the ball drawn from the first urn in the second urn, and place the ball drawn from the second urn in the first urn. Let X_n denote the state of the system after the n^{th} step. Find the transition matrix for this Markov chain.

(4 marks) (c) Determine the stationary distribution for the chain. (If you determine it by guessing, be sure to verify that your guess is correct.)

(1 marks) (d) Suppose that initially urn number 1 contains m black balls. How many steps, on average, will it take until the next time it contains m black balls?

Table 1: Common Distributions

Distribution	Mean	Variance	Characteristic function
Binomial (n, p)	np	$np(1-p)$	$(1-p+pe^{it})^n$
Geometric (p)	$1/p$	$\frac{1-p}{p^2}$	$\frac{pe^{it}}{1-(1-p)e^{it}}$
Poisson (λ)	λ	λ	$e^{\lambda(e^{it}-1)}$
Uniform (a, b)	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{ita}-e^{itb}}{it(b-a)}$
Exponential (λ)	$1/\lambda$	$1/\lambda^2$	$\frac{\lambda}{\lambda-it}$
Normal (μ, σ^2)	μ	σ^2	$e^{i\mu t-\sigma^2 t^2/2}$

Table 2: Cumulative distribution function $\Phi(x)$ of standard Normal distribution

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990